A Spectrum Analysis of Seasonal Adjustment

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Preface

This report was originally published in 1964 as Research Memorandum No. 64 of the Econometric Research Program at Princeton. Shortly after completing this work I wrote another paper ("A Non-Linear Analysis of Seasonal Adjustment," *Business and Economics Section, Proc. ASA,* pp. 196–199, 1964) which mainly summarized these results.

This report also appeared as Chapter 24 in *Essays in Mathematical Economics in Honor of Oskar Morgenstern,* PUP, 1967p. 427[29]. It is substantially the same as the original RM 64.

In addition in this version I have added an Appendix C which contains the programs that were used for the computation of the "rational-function" adjustment method. The listed programs have only been modified to make them compile and execute in current computing environments. This is more fully explained in Appendix C.

All of the Figures were generated using the Princeton IBM 7094 which provided an online graphics display and recording system. Display was provided by an online CRT, and recording used a 35mm microfilm camera. For this version of the paper, the Figures were scanned to PDF from prints from the microfilm. Under suitable conditions, the online display could be used as an interactive graphical output device. Analysis in this mode, which was unusual in 1964, made this work possible.

The source code for the programs and examples of their use are available as explained in the Appendix.

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1. INTRODUCTION

The problem of dealing with the apparent seasonal variation in time series has been treated by a large number of research workers (p. 426[15] Kuznets, p. 427[20] O.E.E.C., and p. 427[27] Wald). One of the most interesting of these treatments has been that of Abraham Wald (p. 427[27] Wald). On the basis of criticisms of the then current methods put forward by Oskar Morgenstern, Wald first analyzed the statistical basis of the most important of these methods and then developed a new method. This method has received relatively little attention and almost no practical utilization (p. 427[20] O.E.E.C., pp. 151-176]). Therefore, we analyze this method and present its derivation (Appendix A). In addition we treat the most widely used method, the Census Method, along with a newer method put forward by Hannan (p. 426[10] Hannan). Finally, we treat a fourth method which is developed on the basis of the analysis of the three previously mentioned methods.

This paper is principally concerned with the analysis of the comparative performance of various methods of seasonal adjustment. The approach taken is to generate the components of an additive seasonal model:

$$Y_t = C_t + S_t + I_t$$

where

Y_t – observed time series

C_t – "trend-cycle" components

S_t – "seasonal" components

I_t – "irregular" components.

The "trend-cycle" plus "irregular" $(C_t + I_t)$ and the "seasonal" (S_t) components are generated separately in order that each seasonal adjustment method may be tested in terms of its effects upon a process, the components of which are known separately. The experimental technique of actually generating series and applying the seasonal adjustment methods to them was adopted partly because of the analytic difficulty of fully analyzing the properties of these methods, some of which are intended to adjust non-stationary series. However it was also felt that this approach would help to give some further insight into the analytic problems of seasonal adjustment. It may also contribute in some way to a more direct understanding of the possible effects of seasonal adjustment as commonly applied to actual data of unknown composition.

In order to discuss the significance of the results of this analysis, it is first necessary to discuss the rationale of seasonal adjustment.

1.1 THE OBJECTIVES OF SEASONAL ADJUSTMENT

A heuristic statement of the objective of seasonal adjustment might be given thus: The elimination of variations in a time series which are attributable to predictable seasonal events in order to display more clearly the more important underlying variations. This statement, while not a rigorous definition, has two interesting implications. The first is that the seasonal variations are predictable and not of interest to analysis of the underlying system. The second is that these seasonal variations are separable from the rest of the series. The acceptance of these implications is a matter of open debate. Their acceptance is normally related to the use to which the data are to be put. If the data are to be used for the estimation of econometric models, it is normally assumed that the seasonal variation is of interest and its contribution to the explanation of the variation of the dependent variables is provided for through the use of seasonal variables and the application of the model to unadjusted data. If on the other hand the data are to be used for single time series extrapolation, one would be indifferent, within the context of linear theory, as to whether the seasonal variables were removed and treated separately or left in the data.

Finally, if the data are to be used to present a time series which "best" represents a realization from some underlying process (which does not involve strictly periodic terms of period one year) for purposes of, say, government policy decisions, then seasonal adjustment may be justified.

It is from this last viewpoint that we approach the problem of determining criteria for measuring the performance of seasonal adjustment methods. In a verbal form our criterion, for the methods may thus be stated loosely as follows: That method is judged "best" which, when it operates on a time series composed of "seasonal" and other variations, produces

a series which most closely approximates the other variations in the original series. This criterion will be more accurately specified and elaborated upon in Section 5.0. However, a difficulty with this criterion should be introduced at the point. If we consider the spectrum of a seasonally adjusted series, it is intuitively clear that the subjective significance which may be attached to contributions to the spectrum of "errors" in the seasonal adjustment will change considerably depending on the frequency at which the error occurs. It is natural to suppose that "errors" introduced by seasonal adjustment which occur at low frequencies would be considered as more undesirable than such "errors" occurring at high frequencies. Underlying this problem is the idea that information at certain frequencies is more important than information at certain other frequencies.

1.2 SCOPE OF THE METHOD OF ANALYSIS

Since we base our analysis on the analysis of spectra, we are restricted to the analysis of the linear information in the time series and linear dependence between two series. This restriction prevents the full analysis of non-linear operators. However, the severity of this restriction is reduced by the fact that nonlinear operators may be, at least partially, analyzed in terms of their effects on linear information and by the fact that in some cases transformations of the processes involved may yield linear relationships.

Another restriction is that of stationary. Strictly, the spectrum estimates used only have meaning for second-order stationary processes. However, spectrum methods have been shown to give useful estimates under a, fairly wide range of non-stationary conditions (p. 426[12] Hatanaka). Thus we assume that our estimates have meaning if the time-averaging property of the pseudospectrum as described in (p. 426[12] Hatanaka) is kept in mind.

2. THE GENERATING PROCESS FOR THE TIME SERIES USED

The underlying process for the time series used is defined by a second order autoregressive scheme of the following form:

$$y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \epsilon_t \tag{2.1}$$

where α_1, α_2 – parameters

ϵ_t – normally distributed random independent numbers.

This corresponds to the conventional trend-cycle and irregular components of the seasonal adjustment model. The coefficients of the autoregressive scheme may be chosen to yield time series which, in terms of spectrum analysis, appear similar to many economic time series. It has been found in the course of various analyses of economic data that the series may frequently be well represented by a low-order autoregressive. It is most commonly found that a first or second-order autoregressive may be used to approximate the estimated spectra quite closely. Interesting discussions of this point occur in (p. 426[8] Granger) and p. 427[21] Parzen), where it is mentioned that a characteristic root of the estimated autoregressive is frequently very close to unity This point has further implications both in terms of the implied stability of the generating process and in terms of the sampling variance of the estimates of the process. The seasonal component was generated by one of the two following processes:

The first process is defined by the following equations:

$$S_{i,k} = S_{1,k}(1 + v_{12i+k}) \quad k = 1, \ldots, 12; \quad i = 1, \ldots, l \tag{2.2}$$

$$v_j = v_{j-1} + \epsilon_j \tag{2.3}$$

where $S_{i,k}$ –seasonal factor of kth month of ith year

v_j – disturbance term

ϵ_j – normally distributed random numbers.

This process produces a seasonal component with a constant pattern (given by $S_{1,k}$) but with a changing amplitude. The amplitude is determined by the first-order autoregressive scheme given by equation (2.3). The variance of the random term ϵ_j determines the amplitude of the variations in the seasonal amplitude.

The second process is defined by the equations:

$$S_{i,k} = \beta S_{i-1,k} + \rho_{i,k} \qquad k = 1, \ldots, 12; \quad i = 1, \ldots, l \tag{2.4}$$

$$\rho_{i,k} = \rho_{i-1,k} + \sum_{j=1}^{12} C_j \ \epsilon_{12i+j+k} \tag{2.5}$$

where:

$S_{i,k}$ –seasonal factor of kth month of ith year,

$\rho_{i,k}$ – disturbance term,

ϵ_n – normally distributed random numbers,

C_j, β – parameters.

Seasonal series generated from this process vary both in amplitude and pattern. Clearly the values for the ϵ_n coefficients determine the amount of correlation between a change in one month's seasonal and a change in the seasonal of neighboring months. At one extreme one may have simply twelve independent first-order autoregressive processes for the twelve monthly factors; or, on the other hand, one may have a high degree of correlation between months so that the seasonal pattern changes very little.

It is felt that these processes fairly describe a large class of possible seasonal components. In particular the first process is in close agreement with Wald's assumptions, while the second seems to fit the assumptions of the Census Method. The assumptions of Wald's method are discussed below (Section 4.2 and Appendix A). The Census Method is fully described in (p. 427[20] O.E.E.C.).

3. DIGITAL FILTERING AND EXTRAPOLATION FOR SEASONAL ADJUSTMENT

3.1 DIGITAL FILTERING

It has recently been recognized in the field of seasonal adjustment (p. 426[10] Hannan) that the application of moving averages is an example of the application of a time-invariant linear operator which may be characterized by its transfer function. From this observation stems Hannan's important observation that, since the effects of the operator are completely

described by its transfer function, it is possible to correct for any unwanted effects when the operator is applied to a time series. Hannan specifically points out that the Spencer 15-point formula affects the amplitude of not only very low frequencies but also the seasonal and higher frequencies. Thus the simple estimate of the seasonal amplitude is biased by the attenuation introduced by the Spencer operator. Hannan goes on to show explicitly the calculations required to compensate for this attenuation.

Rather than use simple unweighted moving averages or particular weighted values (such as Spencer's formula) we have, using the spectrum approach, employed a somewhat more general filtering technique to eliminate low frequencies. These techniques have been used in conjunction with the Hannan, Wald, and rational-function methods, which were all programmed by the authors. The Census Method was received in completely programmed form from the Bureau of the Census and no functional modifications were performed on it. (Since the Census Method has been so extensively used for routine adjustment of actual data is may be used as a standard of comparison.)

Inspection of the transfer functions derived from the simple 12-month moving average and the 15-point Spencer formula (Figure 3.1) shows clearly that the latter is to be preferred on the basis of elimination of frequencies below the fundamental seasonal. The fact that it eliminates some of the variance at 1 cycle per year is unimportant because this is easily corrected for in the final estimate. However, the Spencer formula does have the disadvantage that more observations are lost at the end of the series. Thus the problem of producing estimates for the last months of the series when the seasonal is allowed to vary is aggravated. In addition one feels that it would be desirable to have a filter whose properties were, in some sense, optimal for the problem at hand. For the sake of simplicity in handling phase information it was decided to restrict the class of filters that would be considered, for the initial high-pass filtering operation, to symmetric moving averages. This class of filters is, of course, characterized by transfer functions with phase identically equal to zero at all frequencies. On the basis of previous analysis it seemed natural to adopt the minimum mean square error criterion put forward by Parzen (p. 427[22] Parzen).

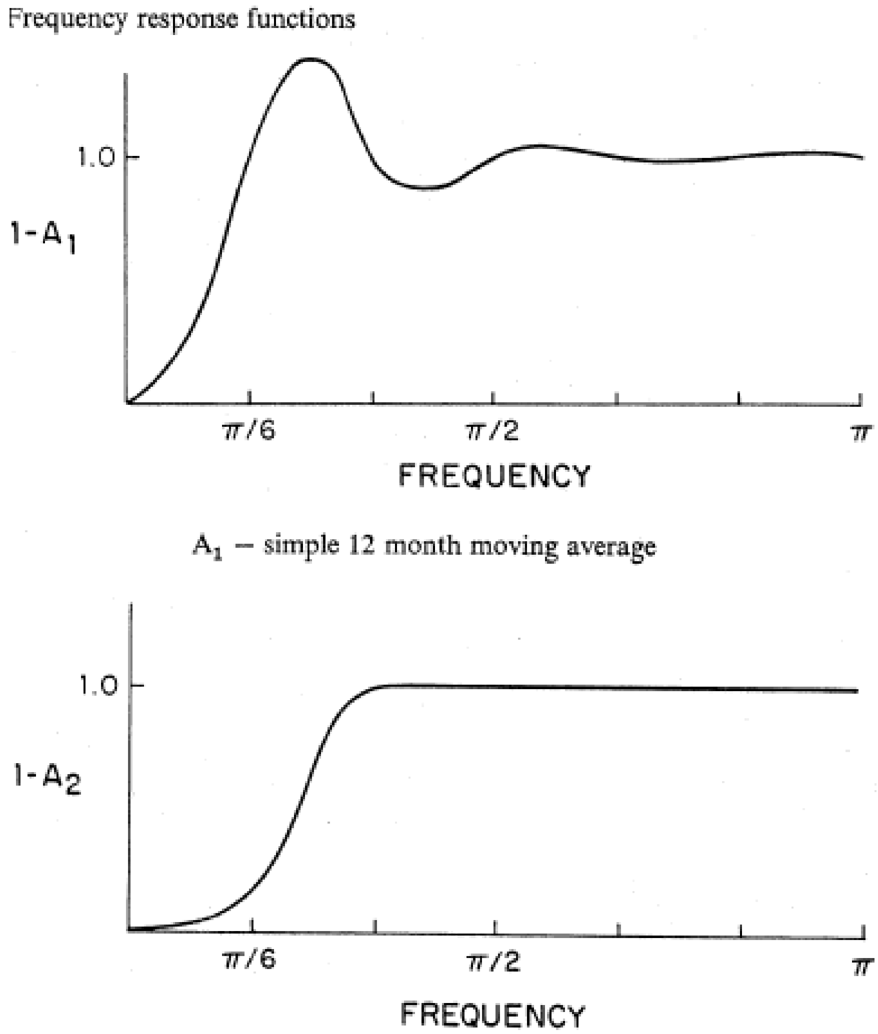


Figure 3.1

Thus we were led to use a filter of the form:

$$\lambda(v) = 1 - 6\Big(\frac{v}{m}\Big)^2 + 6\Big(\Big|\frac{v}{m}\Big|\Big)^3 \qquad 0 \le v \le \frac{m}{2}$$

$$= 2\Big(1 - \frac{v}{m}\Big)^3 \qquad \frac{m}{2} < v \le m \tag{3.1.1}$$

which satisfies this criterion for a specified class of functions. This function is discussed by Parzen (p. 427[22] Parzen) in connection with spectrum windows. It was arbitrarily decided to restrict the filter to a 12-month average. However, it was still possible to vary the bandwidth of the filter to arrive at a "best" value. After some experimentation it was found that the properties of the estimates were not extremely sensitive to small changes in filter bandwidth, and the following transfer function was settled upon:

$$A(\omega) = I - \sum_{v=1}^{m} \frac{\sin kv}{v} \lambda(v) \cos \omega v \tag{3.1.2}$$

where $k = \frac{\pi}{12}$.

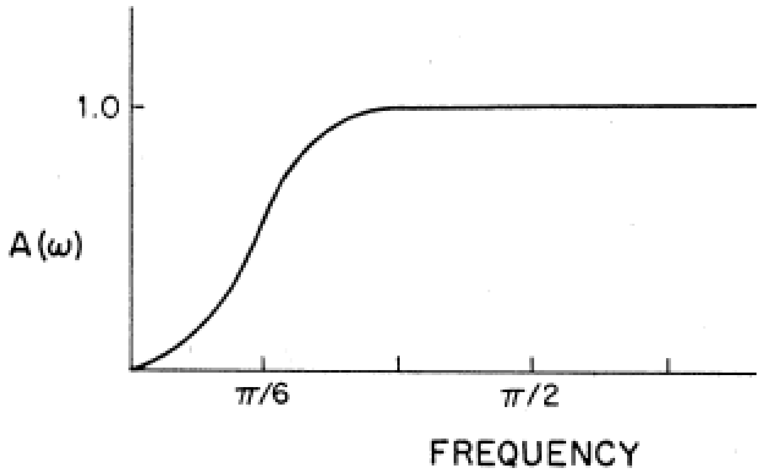


Figure 3.2

The moving average coefficients corresponding to A(w) are obtained by taking the inverse Fourier transform:

$$D_j = \frac{2}{\pi} \sum_{\omega=0}^{\pi} A(\omega) \cos j\omega \qquad j = 0, 1, \ldots, 6 \tag{3.1.3}$$

The gain of this filter is shown in Figure 3.2. The gain is about 0.57 at 1 cycle per year. Very little variation at frequencies below 1 cycle per year will pass through the filtering operation.

3.2 EXTRAPOLATION

After the filtering operation the resultant series is lacking points which correspond to the first and last six observations of the original series. In the context of seasonal adjustment this does not present a problem if only fixed seasonal factors are being estimated. However, if a moving seasonal is being estimated as in the Wald and rational-function methods, then the filtered series must be extended in some way to the last observation of the original series. Wald (p. 427[26] Wald) analyzed this problem in connection with his seasonal adjustment method and incorporated extrapolation of the moving average into the method as it was used in Austria. The extrapolation method is fully described in (p. 427[26] Wald). However, like the monograph on seasonal adjustment (p. 427[27] Wald), this paper has not been translated and no longer seems to be referred to in the literature. Therefore we will briefly outline the technique. The extrapolation method is based upon three assumptions.

Define the 12-month moving average, $\psi(t)$, of a set of observations $\Phi(t)$ by:

$$\psi_{i,k} = \frac{1}{12} \sum_{j=k-6}^{k+5} \Phi_{i,j} + \frac{1}{2}(\Phi_{i,k-6} + \Phi_{i,k+6}). \tag{3.2.1}$$

Defining $s(t)$ as the seasonal component and $z(t)$ as the random component of $\Phi(t)$ we may define a function $f(t)$ by:

$$f(t) = \Phi(t) - s(t) - z(t).$$

The first assumption is then that:

$$\frac{1}{2l+1}\sum_{j=k-1}^{k+l} f_{i,j} \approx \psi_{i,k} \qquad l = 2, 3, 4, 5. \tag{3.2.2}$$

The second assumption defines the seasonal component as it has been defined in Wald's seasonal adjustment method (see Section 4.2). Therefore the seasonal is given by:

$$s(t) = \lambda(t)p(t) \tag{3.2.3}$$

where $p(t)$ is a strictly periodic function and $\lambda(t)$ varies only slowly over time.

The third assumption restricts the random term by the following expression:

$$\frac{1}{m}\sum_{j=k+l}^{k+m} z_{i,j} \approx 0. \tag{3.2.4}$$

The values of m for which assumption three may hold will vary with the kind of data being used.

From these assumptions the following expression for the extrapolated values, $\psi^*(t)$, of $\psi(t)$ is derived:

$$\psi^*_{i,k+6-l} = \alpha_{i,l} - \frac{\sum_{j=k-5}^{k+6} |\Phi_{i,j} - \psi_{i,k}|}{\sum_{j=k-5}^{k+6} |\Phi_{i-1,j} - \psi_{i-1,k}|}(\alpha_{i-1,l} - \psi_{i-1,k+6-l}) \tag{3.2.5}$$

where

$$\alpha_{i,l} = \frac{1}{2l+1}\sum_{j=k+6-2l}^{k+6} \Phi_{i,j}.$$

This expression is valid under the three assumptions (equations 3.2.2, 3.2.3, 3.2.4) for $l = 3, 4, 5$, if (3.2.4) is assumed valid for $m \geq 7$; and is valid for $l = 2, 3, 4, 5$ if (3.2.4) holds for $m \geq 5$. For $l = 0, 1$ and possibly 2 this equation cannot be used. To arrive at estimates for these values of l Wald applies simple linear extrapolation of the last values of the series $\psi(t)$ and the extrapolated values given by equation (3.2.5). In the case of (3.2.4) being taken to be valid for $m = 7$ this leads to extrapolation using the five values

$$\psi_{i,k-1}, \psi_{i,k}, \psi^*_{i,k+1}, \psi^*_{i,k+2}, \psi^*_{i,k+3}.$$

Thus, defining

$$\Delta = \frac{2(\psi^*_{i,k+3} - \psi_{i,k-1}) + (\psi^*_{i,k+2} - \psi_{i,k})}{10} \tag{3.2.6}$$

and $\mu =$ the arithmetic mean of the five values, the last three values of $\psi^*(t)$ are determined by:

$$\begin{aligned} \psi^*_{i,k+4} &= \mu + 3\Delta \\ \psi^*_{i,k+5} &= \mu + 4\Delta \\ \psi^*_{i,k+6} &= \mu + 5\Delta. \end{aligned} \tag{3.2.7}$$

4. DESCRIPTION OF THE SEASONAL ADJUSTMENT METHODS

4.1 HANNAN'S METHOD

The first method that has been examined is the one developed by Hannan (p. 426 [10] Hannan). Here, only its main features will be discussed. The basic model is given by the additive relation:

$$y(t) = p(t) + s(t) + x(t)$$

where $p(t)$ –the trend-cycle component

$s(t)$ –the seasonal component

$x(t)$ –the residual.

x(t) is assumed to be stationary, though this is not essential for the method.

The seasonal component is assumed to be unchanging and of the form:

$$s(t) = \sum_{k=1}^{6} [\alpha(k) \cos \omega_k t + \beta(k) \sin \omega_k t], \qquad \omega_k = \frac{2\pi k}{12} \tag{4.1.1}$$

To remove the low-frequency components (trend-cycle) from the series, Hannan, using the spectrum approach, considers several well-known operators. Using the notation $I - A$ (where A is a moving average operator) for an operator which removes low frequencies, the trend-cycle removed series, $y'(t)$, is given by:

$$y'(t) = [I - A]y(t). \tag{4.1.2}$$

The preliminary estimates of the seasonal $\mu'(j)$ for $j = 1, 2, \ldots, 12$ are derived by calculating the mean for each calendar month:

$$\mu'(j) = \frac{1}{m} \sum_{t=1}^{m} y'_{12t+j} \quad \text{for} \ \ j = 1, 2, \ldots, 12 \tag{4.1.3}$$

where m equals the number of (full) years for which the series $y'(t)$ is computed.

The 12 $\mu'(j)$'s are then adjusted to add to zero by subtracting their mean $\bar{\mu}$; the new estimates are called $\mu(j)$.

The final seasonal adjustments $\hat{a}(j)$ for $j = 1, 2, \ldots, 12$ are then estimated by taking a moving average, with weights $v(k)$, of the $\mu(k)$:

$$\hat{a}(f) = \sum_{k=1}^{12} \mu(k) v(k - j) \qquad j = 1, 2, \ldots, 12 \tag{4.1.4}$$

where $v(k) = v(k + 12)$ for $k \leq O$. $v(k)$ is given by:

$$v(k) = \frac{1}{12} \sum_{s=1}^{11} \frac{1}{1 - h(\omega_s)} e^{-is\omega_k} \tag{4.1.5}$$

where $h(\omega)$ = transfer function of the operator A for frequency ω defined from 0 to π.

The function $v(k)$ is the transformed inverse of the transfer function of the operator $I - A$. Thus the application of $v(k)$ to the $\mu'(k)$ corrects the seasonal weights for any change in the amplitude of variation of the original series at the seasonal frequencies which the operator $I - A$ may have introduced.

4.2 WALD'S METHOD

The second seasonal adjustment method that has been analyzed is the one developed by Abraham Wald in 1936. Wald was at that time associated with the Austrian Institute for Business Cycle Research, under the direction of Oskar Morgenstern. The method was published as Contribution No.9 of that Institute, under the title *Berechnung und Ausschaltung von Saisonschwankungen* (p. 427 [27] Wald). There does not seem to be an English translation of this monograph,[1], which might help to explain why this method is not well known in English-speaking countries. The method is also known as the moving-amplitude method and has been characterized as being able to produce better results than other methods when the seasonal amplitude is thought to change relatively rapidly from year to year. Rapid amplitude changes have been noted in a certain number of agricultural crop series O.E.E.C (p. 427 [20] O.E.E.C, p. 64).

The assumptions of Wald's method require that the seasonal pattern (i.e., the proportionality relationship between the seasonal at each month and the seasonal at each other month) remain constant over time. This constant pattern is used to estimate changes in the amplitude of the seasonal. This approach permits the estimation of more rapid changes in the seasonal amplitude than would be possible if no assumption about the stability of the pattern were made.

Because Wald's method does not appear to be well known, we will give an outline of it here; a full description of the derivation is given in Appendix A. The technique which Wald developed after the publication of (p. 427 [27] Wald) for extrapolation of the moving-average series has been discussed separately in Section 3.2. This extrapolation method was incorporated in the adjustment method as it was employed in Vienna in order to improve the accuracy of the current estimates of the seasonal factors, and has also been used in our computations.

The basic model for Wald's method is that the functions of time which represent seasonal variations, the trend and business cycles, and the random terms, are additive. In Wald's notation

$$\varphi(t) = f(t) + s(t) + z(t) \qquad for \qquad t = 1, 2, \ldots, n \tag{4.2.1}$$

where $\varphi(t)$ –observed monthly series

$f(t)$ –"trend cycle" component

$s(t)$ –seasonal fluctuations

$z(t)$ –residual.

The first step is the removal of the "trend-cycle," $f(t)$. This is accomplished by subtracting the 12-month moving average of $\varphi(t)$ from the original series $\varphi(t)$. This yields the series $\psi(t)$:

$$\psi(t) = \varphi(t) - \varphi^*(t) \qquad for \qquad t = 7, 8, \ldots, (n-6) \tag{4.2.2}$$

where:

$$\varphi^*(t) = \frac{\varphi(t-6) + 2[\varphi(t-5) + \ldots + \varphi(t) + \ldots + \varphi(t+5)] + \varphi(t+6)}{24}$$
$$\text{for} \quad t = 7, 8, \ldots, (n-6). \tag{4.2.3}$$

[1] A very brief summary and application of the method can be found in G. Tintner (p. 427 [24] Tintner)

Replacing $\varphi(t)$ and $\varphi^*(t)$ in this expression by their components (where φ^* indicates the operation of taking the 12-month moving average), we find:

$$\psi(t) = f(t) - f^*(t) + s(t) - s^*(t) + z(t) - z^*(t). \tag{4.2.4}$$

Since $f(t)$ represents the "trend-cycle", we may assume that $f(t)$ can be well approximated by a straight line over periods of 12 months. Therefore $f(t) \approx f^*(t)$.

With respect to the seasonal fluctuation $s(t)$, Wald rejects the assumption that it is merely a 12-month periodic function. The hypothesis that it is a periodic function, which is multiplicative with the original observations $\varphi(t)$ or the "trend-cycle" $f(t)$ is also rejected. Thus the models:

$$s(t) = p(t)\varphi(t); \qquad s(t) = p(t)f(t)$$

or

$$s(t) = p(t)\varphi(t) + q(t)$$

where $p(t)$ and $q(t)$ are 12-month periodic functions, are all considered to be unsatisfactory. Wald instead assumes that $s(t) = \lambda(t)p(t)$. $\lambda(t)$ is an arbitrary function, the value of which will slowly change over time, and $p(t)$ is a 12-month periodic function with mean = 0. In other words the intensity (*amplitude*) of the seasonal fluctuations, indicated by the function $\lambda(t)$, changes slowly with time, but is not systematically related to other variations in the series. The pattern of the seasonal fluctuations, indicated by the function $p(t)$, is however assumed to remain constant over time. The allowance for change in the intensity of the seasonal fluctuations is based on the observation that this intensity is influenced by the trend and the business cycle. However, since there is no a priori reason to expect that this influence follows a well-defined scheme, e.g., that the intensity of the seasonal fluctuations is proportional to the trend, the function $\lambda(t)$ is left arbitrary.

On the basis of the model $s(t) = \lambda(t)p(t)$ it is observed that $|s^*(t)|$, the absolute value of $s^*(t)$, will be smaller, the smaller the fluctuation of $\lambda(t)$ within the period of 12 months, and that $s^*(t) = 0$ if $\lambda(t)$ is constant. This leads to the assumption that $s^*(t) \approx 0$. Equation (4.2.4) is now reduced to:

$$\psi(t) \approx s(t) + y(t) \quad \text{for} \quad t = 7, 8, \ldots, (n-6) \tag{4.2.5}$$

where $y(t) = z(t) - z^*(t)$.

It is convenient at this point to introduce the matrix $\psi(i,k)$, the (i,k)th element of which refers to the kth month of the ith year. Let the corresponding values of $s(t)$ and $y(t)$ be similarly arranged in two matrices, the elements of which will be designated $s(i,k)$ and $y(i,k)$. Computing now the arithmetic mean of the values of the k-month of $\psi(i,k)$ as well as of $s(i,k)$ and $y(i,k)$ one obtains, after substituting $\lambda(i,k)p(i,k)$ for $s(i,k)$ in equation (4.2.4):

$$\frac{1}{m}\sum_{i=1}^{m}\psi(i,k) \approx \frac{1}{m}\sum_{i=1}^{m}\lambda(i,k)p(i,k) + \frac{1}{m}\sum_{i=1}^{m}y(i,k) \qquad \text{for} \quad k = 1, 2, \ldots, 12 \tag{4.2.6}$$

where

$$m = \frac{n}{12} - 1.$$

From these assumptions Wald arrives at the following expression for the estimated seasonal:

$$s(i,k) = a(k)\frac{\sum_{j=k-6}^{k+5}\psi(i,j)a(j)}{\sum_{l=1}^{12}[a(l)]^2}$$

where $a(k) = \frac{1}{m}\sum_{i=1}^{m}\psi(i,k)$ for the m by 12 matrix of seasonal coefficients.

4.3 THE CENSUS METHOD

The third method that has been examined is Census Method II, which will be only briefly described here.[2] In this method the trend-cycle, seasonal and irregular components are assumed to be combined *multiplicatively.* It is sometimes indicated that this is the commonest form of seasonal relationship for the broad mass of economic time series (p. 427 [20] O.E.E.C., p. 58]). However, this model may be transformed into the additive model by taking logarithms. This transformation may introduce certain problems, particularly with respect to its effect on the distributions of the variables in the model. However, on the basis of computed comparisons of the performance of the Census Method with and without taking logarithms, there was no evidence that, for the kind of series dealt with here, any difficulties would be caused by the logarithmic transformation.

An important characteristic of the so-called "moving seasonality" which is incorporated in the Census Method is that no restriction is placed on the nature of any relationships between the changes in *amplitudes* in successive months. The method can therefore take care of changes in the *pattern* of seasonal variation over successive periods of twelve months as well as changes in amplitude (p. 427 [20] O.E.E.C., p. 259, footnote]). However, the changes in both amplitude and pattern of the seasonal ratios are assumed to be gradual and smooth. The method in its original version was not successful when applied to series with drastic changes in S-I (Seasonal-Irregular) ratios[3] as, for instance, total unemployment; nor could it satisfactorily adjust series with constant seasonal patterns but sharply varying amplitudes as, for example, agricultural stocks and farm employment series. However, later versions of the method contain devices which can take better care of series with extreme S-I ratios than could the original.[4] For this study version X-10 has been used as this was the version which was, according to our information, the most highly developed in early 1963.

4.4 A RATIONAL TRANSFER-FUNCTION METHOD

The purpose of this method has been to develop a simple method, based on spectrum concepts, for comparison with the other methods. The method involves the extension of Wald's method to treat a changing seasonal pattern, and the inclusion of the basic ideas of Hannan's method.

The first step of the method is the conventional one of applying a linear operator to remove low-frequency variations. As with the Hannan and Wald methods the operators and notation used are discussed in Section 3.0.

Next it is assumed that the seasonal pattern may be represented in the form of a set of twelve mixed moving average autoregressive processes with identical coefficients. This is a natural assumption if, for reasons of simplicity, the processes which generate the seasonal

---

[2] For a more elaborate description, the reader is referred to Julius Shiskin's paper, "Tests and Revisions of Bureau of the Census Methods of Seasonal Adjustments," *Bureau of the Census Technical Paper* No.5, November 1960. This paper was incorporated (pp. 79-150) in (p. 427 [20] O.E.E.C.).

[3] Seasonal-irregular ratios are the ratios of the original observations to the 15-term Spencer trend-cycle curve.

[4] Suggestions for modification of the method were given in (p. 427 [20] O.E.E.C., pp. 257-311).

coefficients are taken to be linear. Thus the seasonal pattern coefficients are given by:

$$S_{i,k} = \sum_{j=1}^{n} A_j S_{i-j,k} + \sum_{j=0}^{m} B_j \gamma_{i-j,k} \tag{4.4.1}$$

where $S_{i,k}$ – seasonal pattern coefficient of month k, year i

$\gamma_{i,k}$ – random disturbance term

A_j, B_j – coefficients.

Thus it is natural to attempt to estimate $S_{i,k}$ from the filtered series by:

$$\hat{S}_{i,k} = \sum_{j=1}^{n} A_j \hat{S}_{i-j,k} + \sum_{j=0}^{m} B_j \psi_{i-j,k} \tag{4.4.2}$$

where $\hat{S}_{i,k}$ – estimated seasonal

$\psi_{i,k}$ – filtered series

A_j, B_j – coefficients.

If the Z-transform operator (defined by $Z^{-n} = x_{t-n}$) is applied to the above equation and the terms rearranged we have:

$$\hat{S}_{i,k} = \frac{\sum_{j=0}^{m} B_j Z^{-j}}{1 - \sum_{j=2}^{n} A_j Z^j} \psi_{i,k} \tag{4.4.3}$$

From equation (4.4.3) it is clear that we are simply applying a time invariant, linear, rational function operator to the 12 series $\psi_{i,k} (i = 1, \ldots, l; k = 1, \ldots, 12)$.

The coefficients A_j and B_j might be estimated for each series on the basis of a minimum mean square error criterion. However, in the interest of simplicity and generality the coefficients were in fact determined on the basis of more general, and in part heuristic, criteria. The transfer function defined by equation (4.4.3) should have a gain characteristic which in some sense minimizes the error of the estimate $\hat{S}_{i,k}$. One may assume that $S_{i,k}$ has relatively high spectral density at low frequencies, while the spectrum of $\gamma_{i,k}$ is relatively flat. Then the two spectral densities will be of the form given in Figure 4.4.1.

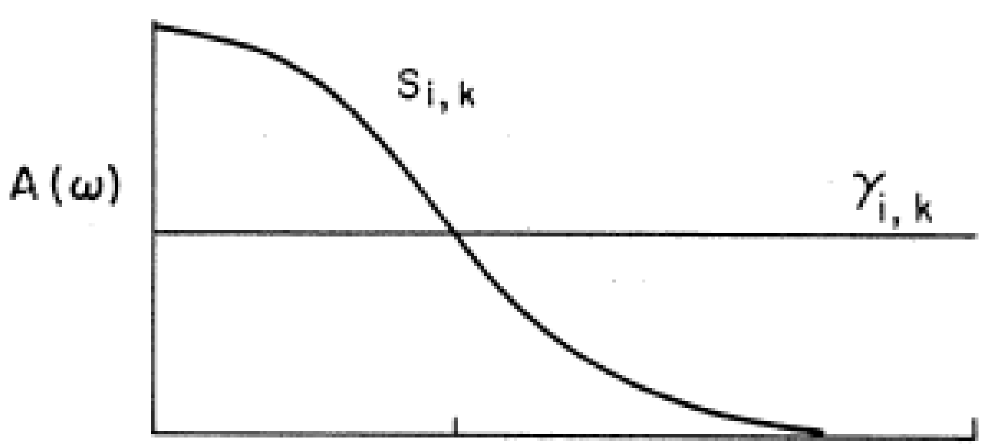


Figure 4.4.1

The transfer function given in equation (4.4.3) should then take the form:[5]

$$A(\omega) = \frac{f_s(\omega)}{f_s(\omega) + f_\gamma(\omega)}$$

where $A(\omega)$ – transfer function of filter

$f_s(\omega)$ – spectrum of seasonal coefficients $S_{i,k}$ $\quad i = 1, \ldots, l$ for all k

$f_\gamma(\omega)$ – spectrum of random term $\gamma_{i,k}$ $\quad i = 1, \ldots, l$ for all k.

Given the above assumptions $A(\omega)$ will take the form indicated in Figure 4.4.2.

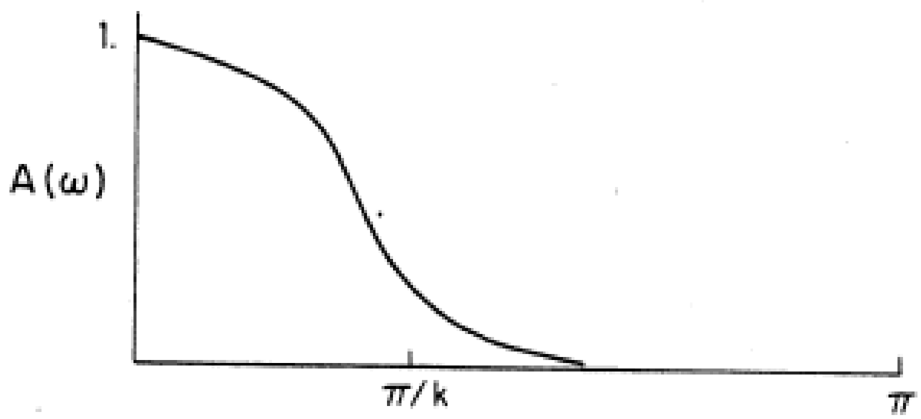


Figure 4.4.2

It seems reasonable for many economic series to assume that π/k falls in the range of about 0.2 to 0.3 cycles per year. This gain characteristic could be approximated by a symmetric moving average filter in a way similar to that applied in the Census Method. However the rational function filter seems more natural, given the assumptions made about the way in which the seasonal variation is generated.

In deciding on the transfer function given in equation (4.4.3) it would be desirable to apply general analytic criteria in terms of both the gain and phase characteristics. However, appropriate general methods based on some practical error-of-estimate concept are not yet available. Therefore the coefficients were simply chosen on the basis of inspection of the transfer function and experimentation.

To this point the method has been analogous to the Census Method in that the seasonal coefficients have been estimated from the twelve annual series. However, we now make the assumption that the seasonal coefficients are intercorrelated. We will, in fact, assume that for relatively high-frequency variations the seasonal factors vary proportionally. This is analogous to Wald's assumption of a constant seasonal pattern. However, as previously assumed, we allow the seasonal pattern (i.e., the proportionality factors) to vary relatively slowly. Thus we attempt to combine the assumption of the Census Method that each seasonal coefficient may change very slowly but independently of the others with the assumption of Wald's method that the amplitude of the seasonal pattern may vary relatively rapidly.

[5] For an exact description of the optimal filter transfer function cf. L. A. Wainstein, p. 427[25] Wainstein.

In order to derive the estimate for the seasonal amplitude we have simply paralleled Wald's derivation. This results in the following expression:

$$\tilde{S}_{i,k} = \frac{\hat{S}_{i,k} \sum_{j=k-5}^{k+6} \hat{S}_{i,j} \psi_{i,j}}{\sum_{j=k-5}^{k+6} (\hat{S}_{i,j})^2}. \tag{4.4.4}$$

To complete the adjustment procedure the estimated seasonal, $\tilde{S}_{i,k}$ is corrected for any bias introduced by the original filtering operation and is then subtracted from the original data.

The actual equations computed are as follows:

First the low frequencies are removed by:

$$\psi_t = \Phi_t - \sum_{j=-6}^{6} D_j \Phi_{t+j}. \tag{4.4.5}$$

The last six values of ψ_t are extrapolated, and the D_j's are determined, as described in Section 3.0.

Then, starting values for the rational function filter are computed by averaging the first four available years of the series ψ_t

$$\hat{S}_{1,k} = \frac{1}{4} \sum_{i=2}^{5} \psi_{i,k} \qquad k = 1, \ldots, 12$$

$$\hat{S}_{2,k} = S_{1,k} \tag{4.4.6}$$

Next the rational function filter is applied according to

$$\hat{S}_{i,k} = A_1 \hat{S}_{i-1,k} + A_2 \hat{S}_{i-2,k} + B_1 \psi_{i,k} + B_2 \psi_{i-1,k} + B_3 \psi_{i-2,k} \quad i = 3, \ldots, l; k = 1, \ldots, 12. \tag{4.4.7}$$

where, for all computations, we used: $A_1 = 1.57, A_2 = -0.7, B_1 = 0.03, B_2 = 0.04, B_3 = 0.06$.[6]

These estimates of the evolving seasonal pattern are then used to estimate the seasonal coefficients according to:

$$\tilde{S}_{i,k} = \frac{\hat{S}_{i,k} \sum_{j=k-5}^{k+6} \hat{S}_{i,j} \psi_{i,j}}{\sum_{j=k-5}^{k+6} (\hat{S}_{i,j})^2} \tag{4.4.8}$$

Finally, the $S_{i,k}$ are corrected by:

$$\tilde{S}'_{i,k} = \sum_{j=1}^{12} \alpha_j \tilde{S}_{i,k-j} \tag{4.4.9}$$

where the α_j's are determined by the inverse of the transfer function of the D_j's given in equation (4.4.5).

The $\tilde{S}_{i,k}$'s are extrapolated to the ends of the series, and the seasonally adjusted series is formed by:

$$\Phi_t^s = \Phi_t - \tilde{S}'_t. \tag{4.4.10}$$

[6] These coefficient values were added after publication.

5. OUTLINE OF COMPUTATIONS

As shown in the flow chart given in Figure 5.1 the typical computation consisted of the following steps:

5.1 The computation of the underlying and seasonal series for a set of specified parameter values.

5.2 The seasonal adjustment of the sum of the underlying series and the seasonal.

5.3 The computation of the spectrum and cross-spectrum properties of the following pairs of series:

5.3.1 The adjusted series and the underlying process.

5.3.2 The adjusted series and the sum of the underlying process and the seasonal.

5.3.3 The adjusted series and the seasonal.

5.3.4 The seasonal series and the difference between the sum of the underlying process and the seasonal on the one hand and the adjusted series on the other. This computation is, then, a comparison of the actual seasonal and the seasonal estimated by the seasonal adjustment method.

On the basis of the general statement that the quality of the seasonal adjustment method may be measured by its ability to accurately recover the underlying process from the sum of the seasonal and this process, the following ideal results may be identified in terms of these calculations. For calculation 5.3.1 the perfect adjustment method would produce spectra with identical shapes and a cross-spectrum with unit gain and zero phase at all frequencies. The coherence should be one at all frequencies. For calculation 5.3.2 the cross-spectrum will depend on the

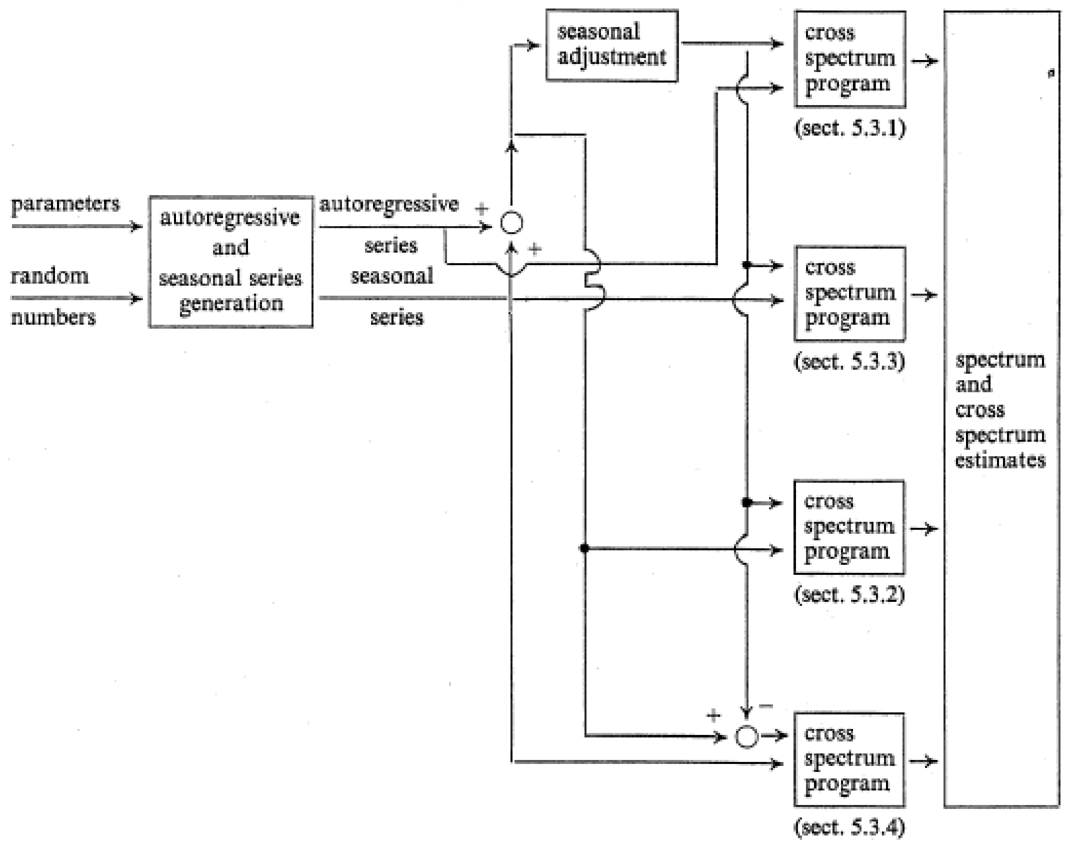


Figure 5.1

seasonal component. The reason for including this calculation is not to provide a direct test of the seasonal adjustment method, but to present the cross-spectrum for comparison with the cross-spectrum which may be computed from actual series with unknown seasonals. The perfect adjustment method would produce for calculation 5.3.3 a coherence consistent with the hypothesis of independence of the series. Therefore, the cross-spectrum should not be significant. Finally, in calculation 5.3.4 the perfect method should produce a unit correlation, as in calculation 5.3.2.

Imperfections in the various methods may show up in a considerable variety of ways as will be discussed in the next section.

6. EXPERIMENTAL RESULTS

In determining the experimental procedure for these computations it was necessary to decide on the values of two sets of parameters. One set involved the values of the autoregressive scheme, while the other was the parameter vector for the seasonal pattern. Through the computation of several pilot runs it was found that the seasonal adjustment methods were not very sensitive to changes in the autoregressive parameter values over a considerable range (i.e., those yielding characteristic periods from about 2 years to ∞).

Therefore the calculations were mainly performed with only one set of values. The values of the parameters used in equation (2.1) are:

$$\alpha_1 = 1.4$$
$$\alpha_2 = -0.40.$$

It is interesting to note that the roots of the characteristic equation of equation (2.1), for these parameter values, are both real with one equal to 1.0 and the other equal to 0.4. Thus, the solution of the equation is not oscillatory. However, the time series generated by this process appear to be better approximations for a wide class of economic time series than series generated using parameter values which give an oscillatory solution.

For the seasonal processes it was, however, found that the methods were sometimes differentially sensitive to the process and parameter values used. Therefore the computations were performed with several sets of values.

The first process, equations (2.2), (2.3), gives a seasonal with a constant pattern. The only variable parameter is the variance of the error term. Several values for the variance were used. This process will be referred to as Type 1.

The second process, equations (2.4), (2.5), allows the seasonal pattern to vary slowly but may preserve some correlation between months. Two sets of values for the C_j were used. The first set is:

$$\begin{gathered} C(1) = 0.6 \\ C(2) = 0.1 \\ C(3) = 0.2 \\ C(4) = 0.05 \\ C(5) = 0.05 \\ C(i) = 0 \quad i = 6, \ldots, 12. \end{gathered}$$

This process will be referred to as Type 2. The second set of values allows each month to vary independently of the other months. These values are:

$$\begin{gathered} C(1) = 1.0 \\ C(i) = 0.0 \quad i = 2, \ldots, 12. \end{gathered}$$

This process will be referred to as Type 3. For each of the sets of values for the vector $C(i)$ several values of the variance of the term ϵ_j were used in order to give several different levels of amplitude change in the seasonal. In addition, values of β other than one were used in order to introduce trends in the seasonal amplitude.

The results of these computations show that all of the methods performed quite well when the generating process conformed to the assumptions on which the particular method was based. In the simple case of the constant seasonal (Type 1 with no error term) Hannan's method was definitely superior. Wald's method performed well for a constant seasonal pattern but changing amplitude. The Census Method performed well when both amplitude and pattern were changing if the changes in amplitude did not become large or relatively rapid. The rational-function method tended to combine the ability of the Census Method to adjust for very slow changes in pattern with the ability of Wald's method to estimate changes in amplitude. As the characteristics of each method are considerably different we will discuss each method separately in greater detail.

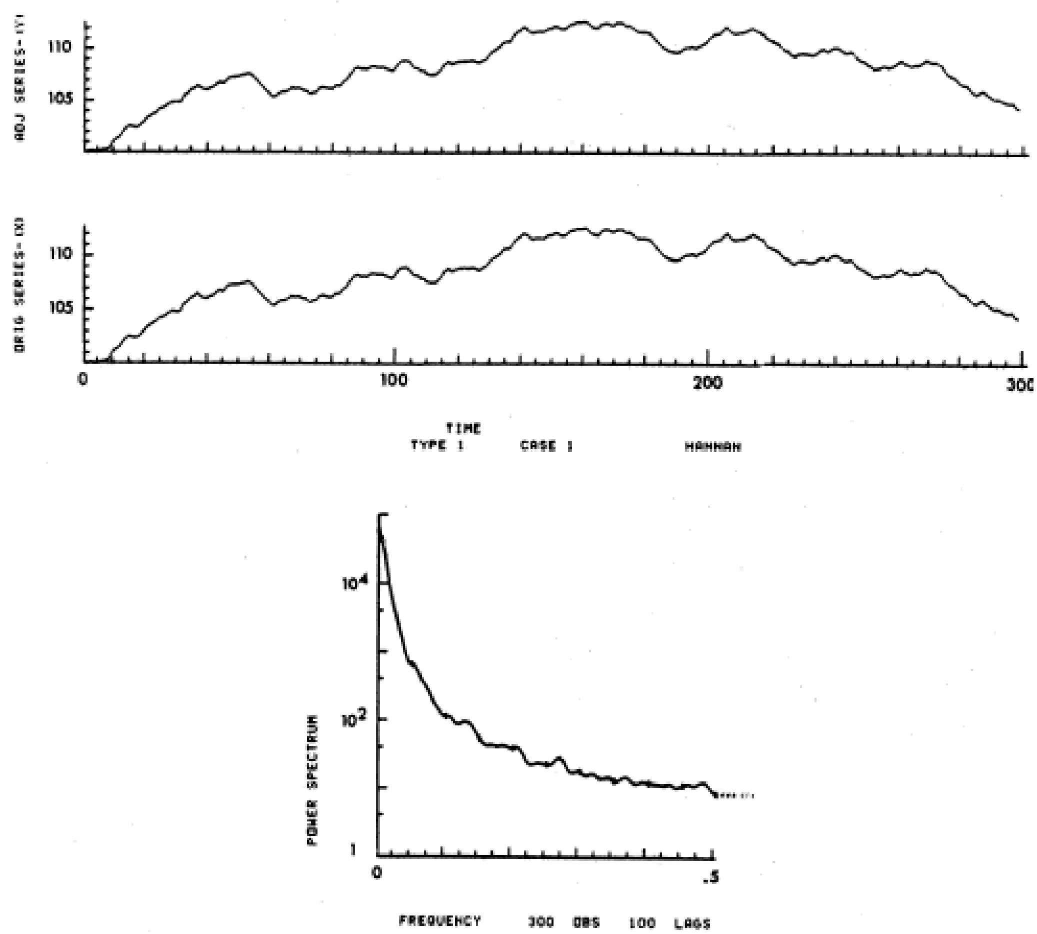
First, however, it is necessary to explain the general form of the graphical results. Each page of graphs (containing either two or four graphs) displays the results of the analysis of two time series. Viewing the page of graphs with the figure label at the bottom, the top graph displays the two time series used in the computation. The graph is divided in half horizontally. The upper half of the graph shows the series resulting from the seasonal adjustment process while the lower half shows the original generated series. The lower row of graphs gives the spectra of the two series and the cross-spectrum between them. The first graph in this row gives the spectra of the two series. The spectrum of the series from the seasonal adjustment process is labeled Y while the other series is labeled X. In some cases, as in Figure 6.01.1, these two spectra lie on top of each other and are not distinguishable. If, in addition, the two series were not significantly different at any frequency in terms of the cross-spectrum, as for Figure 6.01.1, then the graphs of the cross-spectrum are not shown. The deletion of the two graphs following the graph of the spectra implies that the coherence between the two series was not significantly different from unity at any frequency. In the cases where the two series were significantly different the graph following the graph of the spectra gives the coherence between the two series. The last graph shows the transfer function of the two series where the original series is taken as the input and the series from the seasonal adjustment process as the output. The upper half of the graph gives the gain while the lower half displays the phase.

The following index lists each of the figures and gives the two series used in each case as well as the type of seasonal and the name of the seasonal adjustment method.

INDEX TO FIGURES

6.1 HANNAN'S METHOD

As was mentioned above, Hannan's method provides the "best" seasonal adjustment of a series which contains a constant seasonal. The seasonally adjusted series is nearly identical to the original autoregressive series; this is clearly indicated in Figure 6.10. When the seasonal changes, Hannan's method simply takes an arithmetic average of each month's values and estimates this constant average seasonal. Thus Hannan's method continues to perform satisfactorily only as long as the variations in the seasonal can be adequately represented by their time averages.



ADJ SERIES-(Y)
110
105
ORIG SERIES-(X)
110
105
0
100
200
300
TIME
TYPE 1
CASE 1
HANNAN
POWER SPECTRUM
10^4
10^2
1
0
.5
FREQUENCY
300 OBS
100 LAGS

Figure 6.01.1

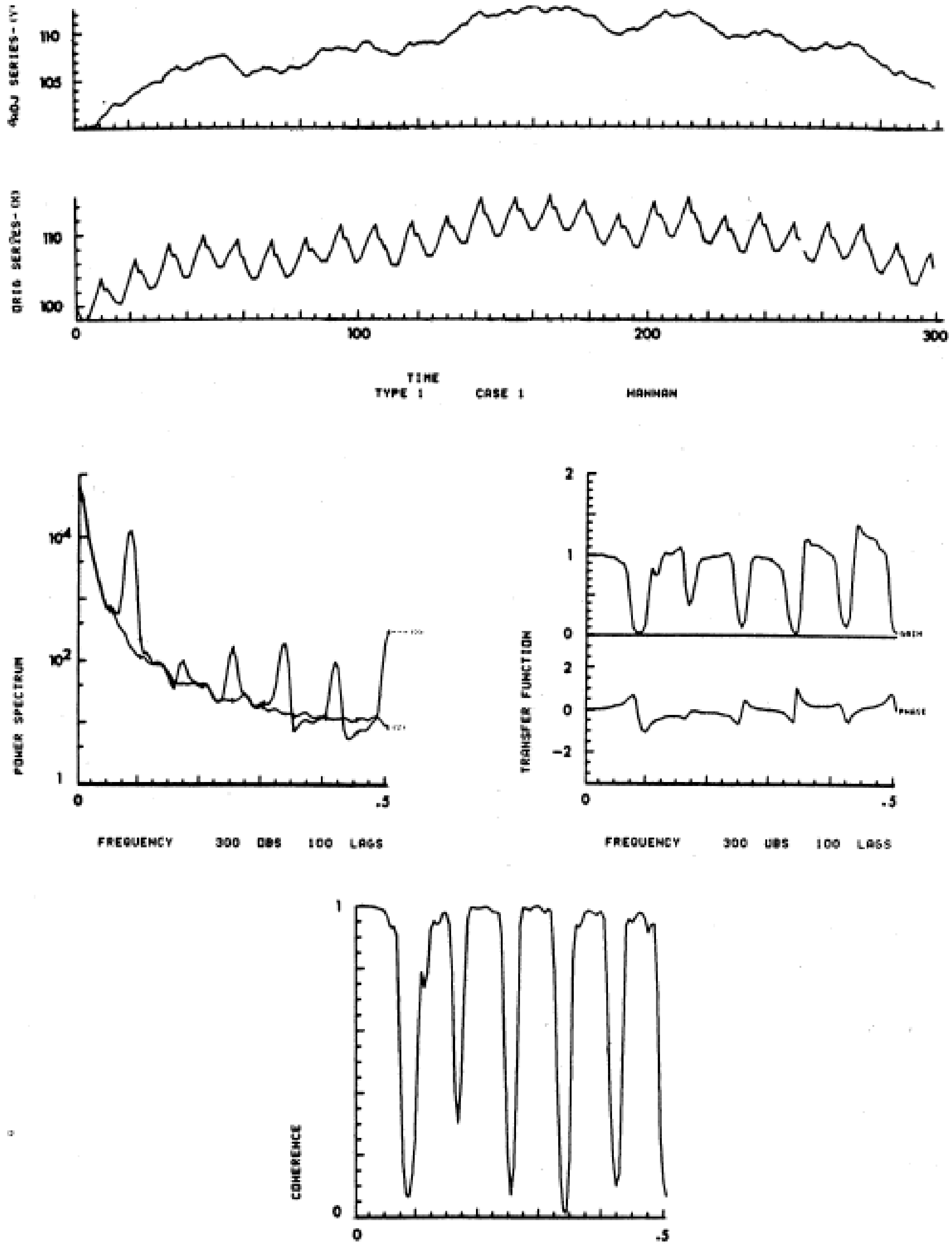

ADJ SERIES-(Y)
110
105
ORIG SERIES-(X)
110
100
0
100
200
300
TIME
TYPE 1 CASE 1 HANNAN
POWER SPECTRUM
10^4
10^2
1
0
.5
FREQUENCY 300 OBS 100 LAGS
TRANSFER FUNCTION
2
1
0
2
0
-2
GAIN
PHASE
0
.5
FREQUENCY 300 OBS 100 LAGS
COHERENCE
1
0
0
.5

Figure 6.01.2

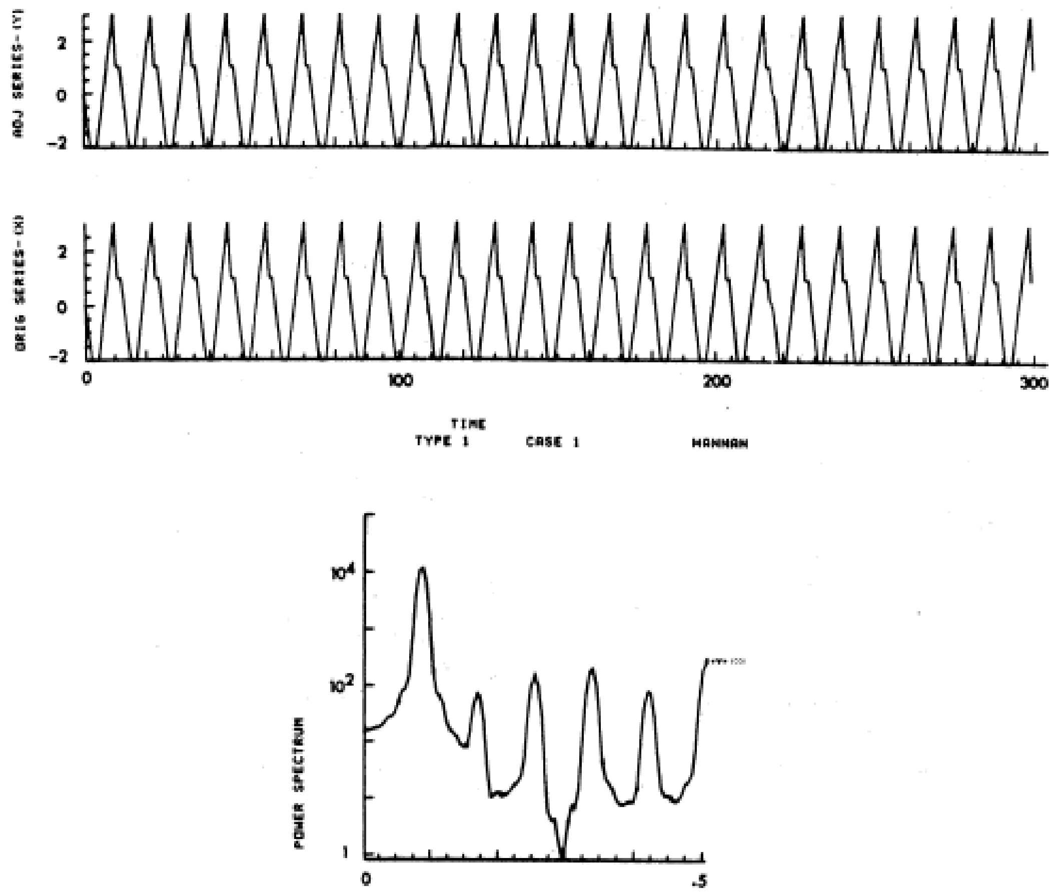


Figure 6.01.3

In this context it is important to mention the effect of the Hannan method on a non-stationary seasonal. To take the simplest case of a linear trend in amplitude, it is clear that the estimated seasonal has a constant amplitude equal to the average amplitude of the actual seasonal. Thus, for the case of a linear trend, the error of the estimate reaches a maximum at both ends of the series and a minimum at the mid-point. In this case the performance indicated by cross-spectrum analysis may also be somewhat misleading due to the time-averaging property of the spectrum estimates. The cross-spectrum shows the average error of the method over the sample. If, however, one is not only interested in the average error but also in the expected error of the estimate for the last observation (or final set of 12 observations) then the cross-spectrum alone does not give a complete measure of performance. As was mentioned previously, this problem, and others relating to non-stationarity, are problems where the concept of the pseudo-spectrum may be applied. It has not been possible in the present paper to pursue this analysis in detail.

The practical limitations of Hannan's method are obvious. First, it is not often felt to be the case that the seasonal can adequately be represented by a set of constant monthly coefficients. Second, the fact that the use of stationarity leads in this case to the result that if the method is sequentially applied from year to year to a set of data to which each

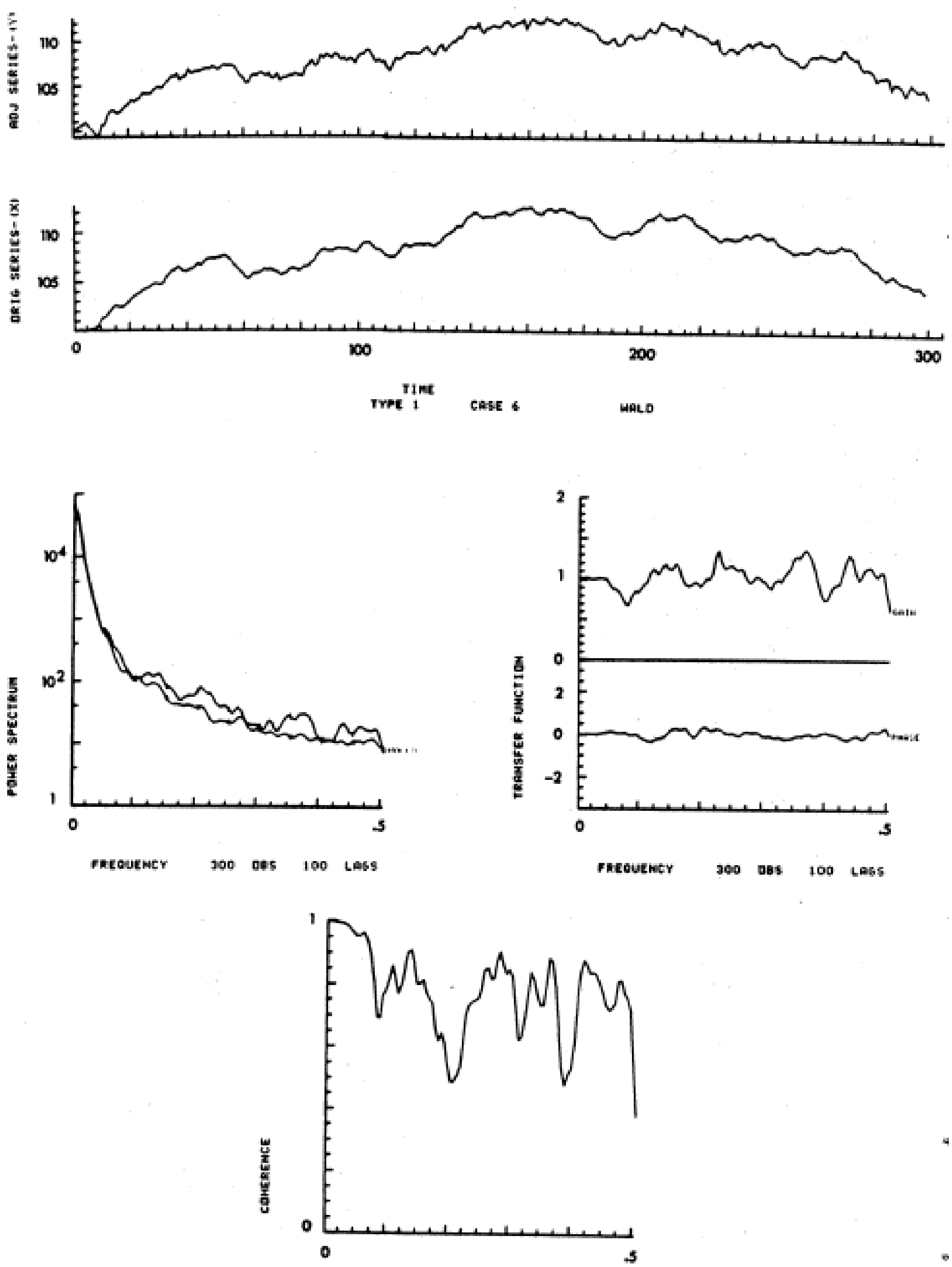
year 12 new observations are added, the seasonally adjusted series will in general have different values for corresponding months, not only in the current or recent years, but from the beginning of the series. As has been frequently observed this leads to the necessity of continuing revisions.

Clearly, the major contribution of Hannan's method is not to be given in terms of its potential application to actual data but rather in its explicit treatment of effects of linear operators in terms of frequency decomposition. Given a constant seasonal, Hannan's technique of adjusting the seasonal factors for any effects of the initial filtering operation leads to better estimates in terms of bias. This technique, obviously, is not restricted to Hannan's method. It is applicable to any method which employs a linear operator before estimation of the seasonal. In fact this adjustment technique, as mentioned previously, was experimentally applied to Wald's method and was used in the rational transfer function method. For a recent development by Hannan, which arrived too late for analysis in the present paper, see (p. 426[11] Hannan).

6.2 WALD'S METHOD

Wald's method, since it employs certain nonlinear operators, affects the linear information in the series even when the actual seasonal is constant. However, these effects are typically quite small and are evident only at relatively high frequencies.

At the low-frequency end of the spectrum, Wald's method produces, for a constant seasonal, an estimate of the autoregressive series very nearly as good as Hannan's method. As is well known, the point at which Wald's method breaks down is when the pattern of the seasonal is allowed to change. Wald's method estimates a constant average pattern (as does Hannan's method) and uses this constant pattern to produce a "best" estimate of the change in amplitude of the seasonal for each year. This approach provides relatively very good estimates of the change in amplitude (particularly for rapid changes in amplitude) of the seasonal when the assumption of the constant pattern is met. In comparing Wald's method with a method which does not rely on the relationship of the seasonal in one month to the seasonal in each other month, it is clear that (again assuming the constant pattern) the use of the information contained in all twelve observations for a given year will lead to a better estimate than the use of only the series of annual observations for each month independently. Figure 6.02 shows the results of the application of Wald's method to a series containing a seasonal of constant pattern but varying amplitude. Figures 6.03 and 6.04 show the results of the use of Wald's methods when both the amplitude and pattern are allowed to change.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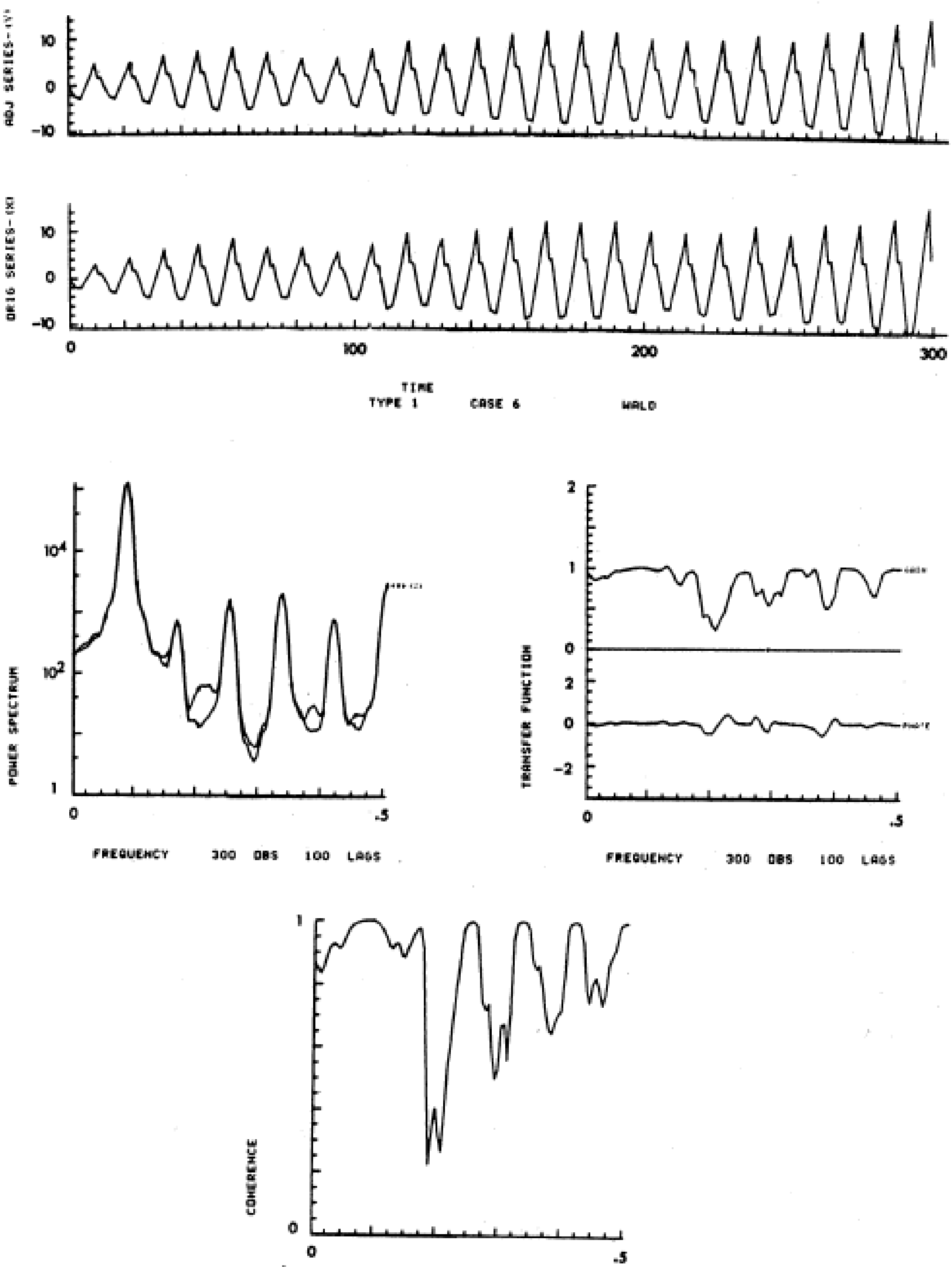
105
ORIG SERIES-(X)
110
105
0
100
200
300
TIME
TYPE 1
CASE 6
WALD
POWER SPECTRUM
10^4
10^2
1
0
.5
FREQUENCY
300 OBS
100 LAGS
TRANSFER FUNCTION
2
1
0
2
0
-2
0
.5
FREQUENCY
300 OBS
100 LAGS
COHERENCE
1
0
0
.5

Figure 6.02.1


ADJ SERIES-(Y)
ORIG SERIES-(X)
10
0
-10
0
100
200
300
TIME
TYPE 1
CASE 6
WALD
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10^2
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0
.5
FREQUENCY
300 OBS
100 LAGS
TRANSFER FUNCTION
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FREQUENCY
300 OBS
100 LAGS
COHERENCE
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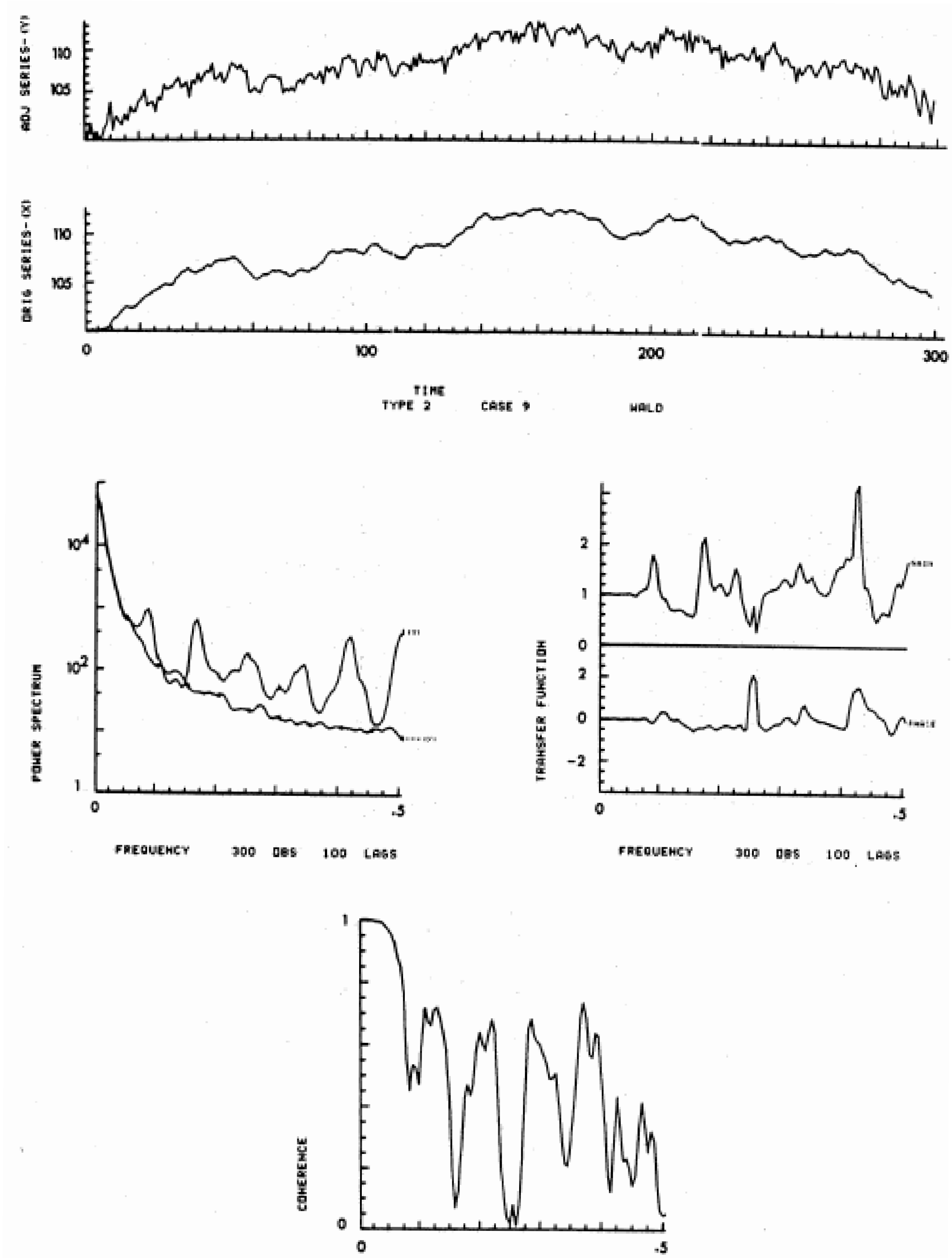
0
.5

Figure 6.02.2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105
ORIG SERIES-(X)
110
105
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100
200
300
TIME
TYPE 2
CASE 9
HALD
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10^2
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.5
FREQUENCY
300 OBS
100 LAGS
TRANSFER FUNCTION
2
1
0
2
0
−2
0
.5
FREQUENCY
300 OBS
100 LAGS
COHERENCE
1
0
0
.5

Figure 6.03.1

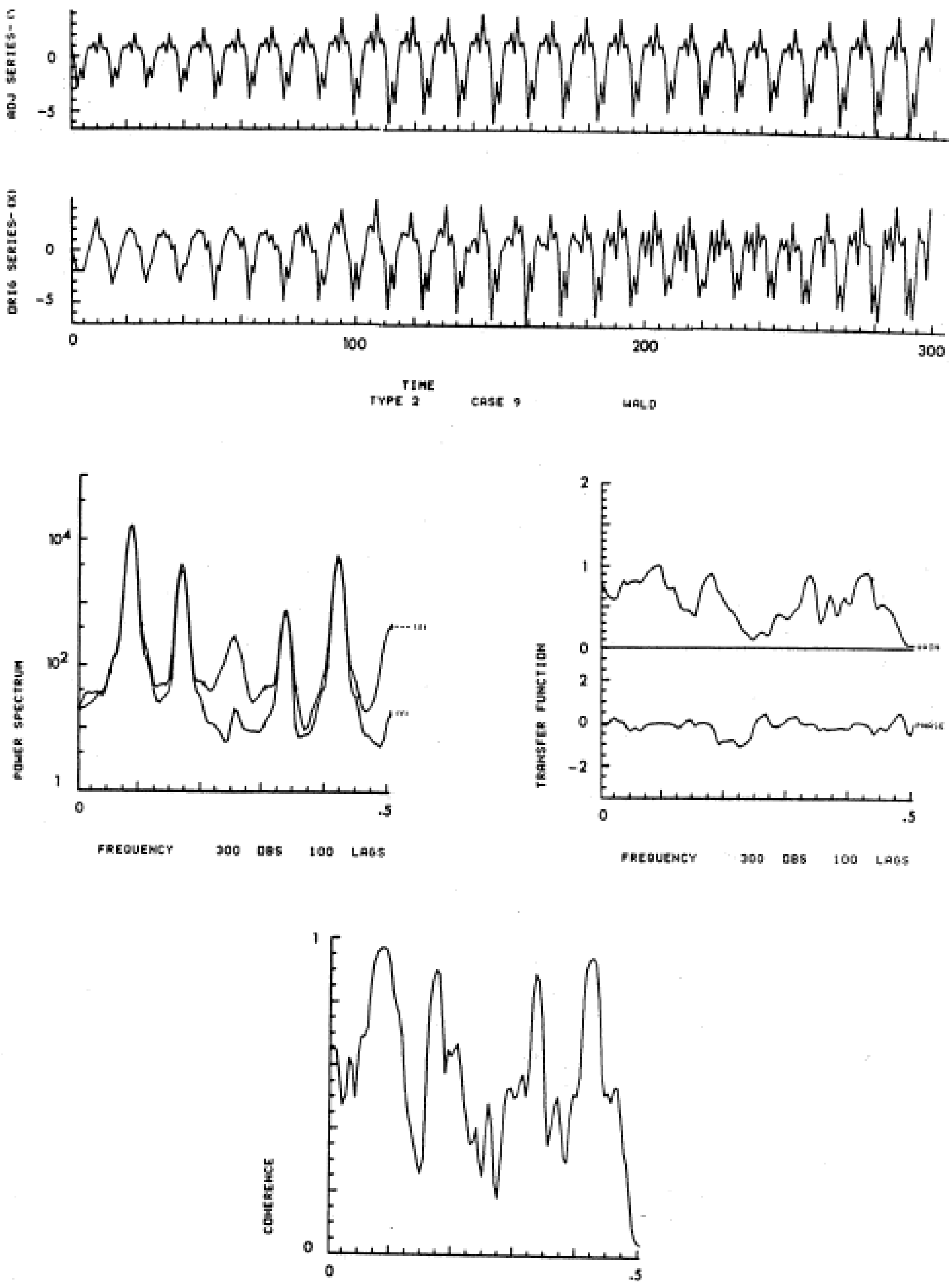
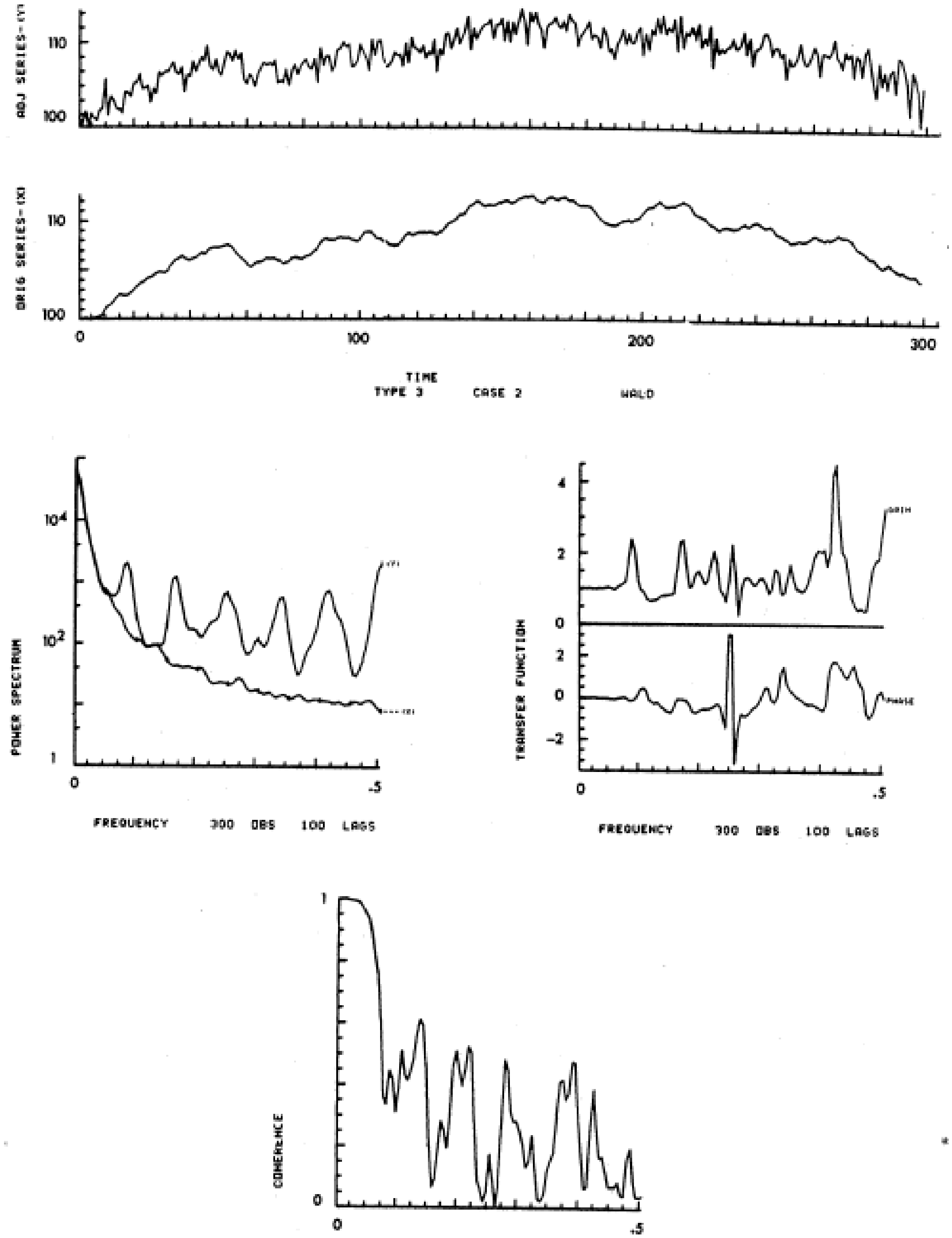

ORIG SERIES-(X)
TIME
TYPE 2
CASE 9
WALD
POWER SPECTRUM
FREQUENCY
300 OBS
100 LAGS
TRANSFER FUNCTION
FREQUENCY
300 OBS
100 LAGS
COHERENCE

Figure 6.03.2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ORIG SERIES-(X)
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100
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100
200
300
TIME
TYPE 3
CASE 2
WALD
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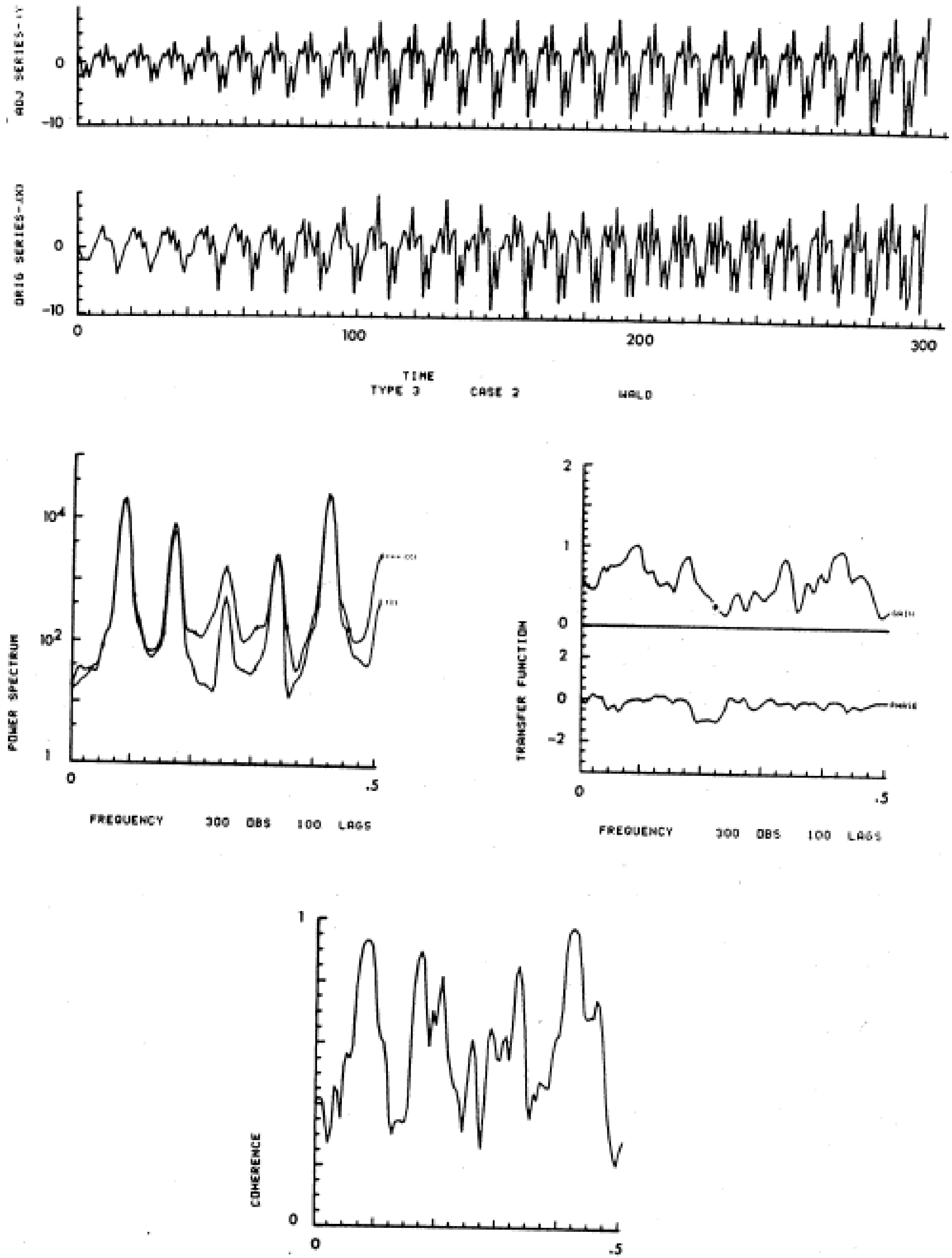
FREQUENCY
300 OBS
100 LAGS
TRANSFER FUNCTION
4
2
0
2
0
-2
0
.5
FREQUENCY
300 OBS
100 LAGS
COHERENCE
1
0
0
.5

Figure 6.04.1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ORIG SERIES-(X)
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0
100
200
300
TIME
TYPE 3
CASE 2
WALD
POWER SPECTRUM
10^4
10^2
1
0
.5
FREQUENCY
300 OBS
100 LAGS
TRANSFER FUNCTION
2
1
0
-2
GAIN
PHASE
COHERENCE

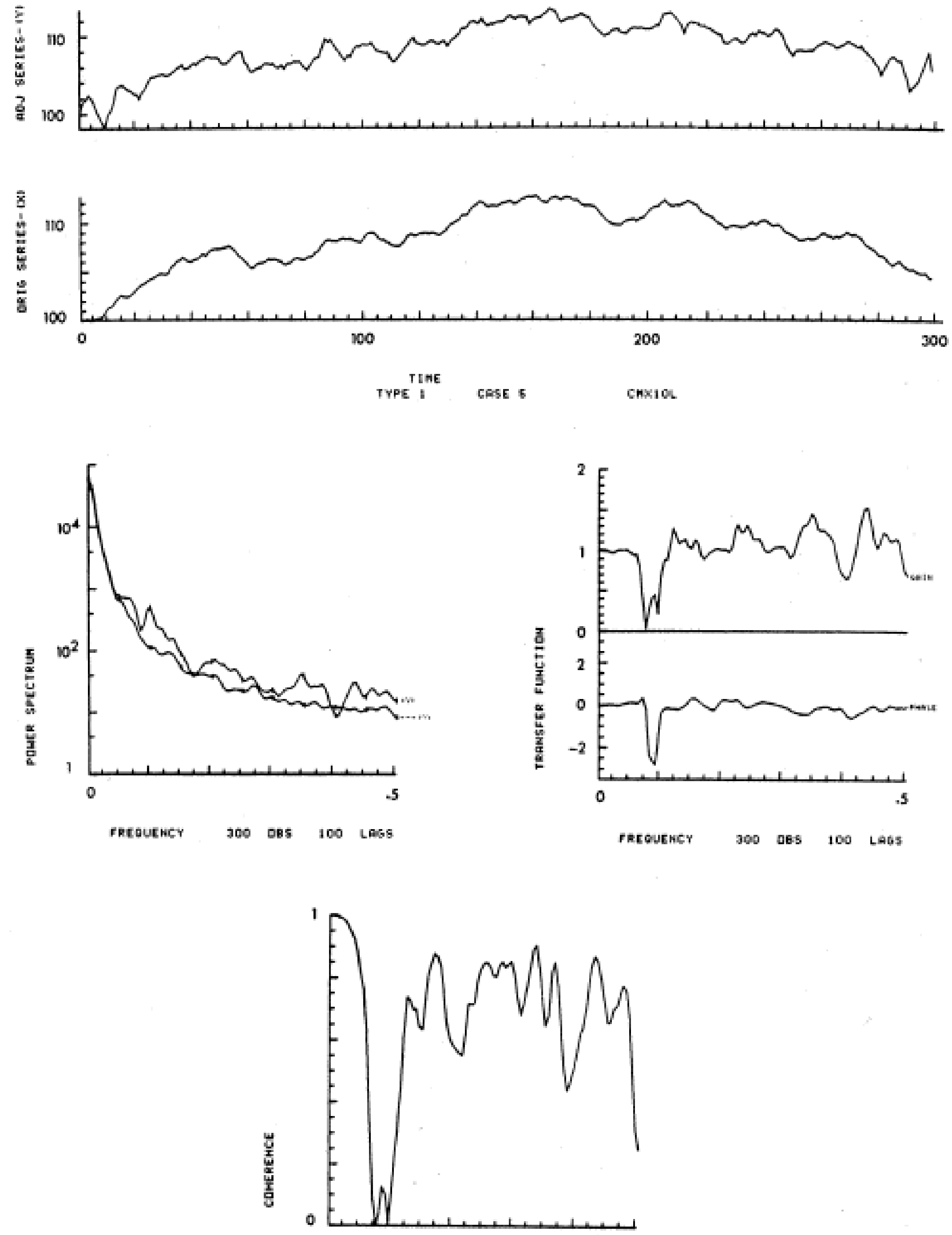
Figure 6.04.2

6.3 THE CENSUS METHOD

The Census Method is more general than either of the two previously discussed methods in that it does not make use of the assumption of either a constant seasonal amplitude or a constant seasonal pattern.

However, there are two respects in which the Census Method appears to be comparatively inferior to the two methods above. First no use is made of Hannan's technique of correcting for the bias introduced by the application of a linear operator to the original data. Since the Census Method uses the Spencer 15-point formula for this operation, the 12-month component is attenuated to 20% of its original value. This bias is not clearly evident in the results because Spencer's formula is applied only after most of the seasonal has been removed using a simple moving average. The use of Hannan's correction procedure eliminates the need for a two-(or more) stage process as used in the Census Method. The second weakness of the Census Method is its poor response, when compared to Wald's method, to relatively high-frequency variations of the seasonal amplitude. Part of the reason for this relatively poor frequency response characteristic is the large number of observations for a single month which are required in order to form a stable estimate (in the sense that the estimate contains very little spectrum power at high frequencies) of each monthly factor.

The observation of overriding importance that must be made about the Census Method is, however, that it performs reasonably well under a wide range of conditions. Only when the seasonal amplitude changes very rapidly is the seasonal not completely removed, at least in the sense of the removal of the peaks in the estimated spectrum. However, the method does allow sharp changes in the seasonal to appear in the adjusted series. In addition, under certain conditions, the method alters the series at frequencies other than the seasonal. Thus the coherence between the adjusted and the autoregressive series is sometimes quite low at low (but non-seasonal) frequencies. The gain is also sometimes considerably different from 1 at low frequencies. However, the phase is uniformly close to zero even under extreme conditions. The fact that the phase of the autoregressive series is not significantly altered by the seasonal adjustment method is of greatest importance for the interpretation and use of the adjusted series. Since the series may be used by the government for stabilization policy measures, it may be important to the stability of the system that a phase lag not be introduced. Figure 6.05 shows the performance of the Census Method for a constant seasonal pattern, while Figures 6.06 and 6.07 show the result of the same method when both the amplitude and pattern of the seasonal change considerably.


ADJ SERIES- (Y)
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100
ORIG SERIES- (X)
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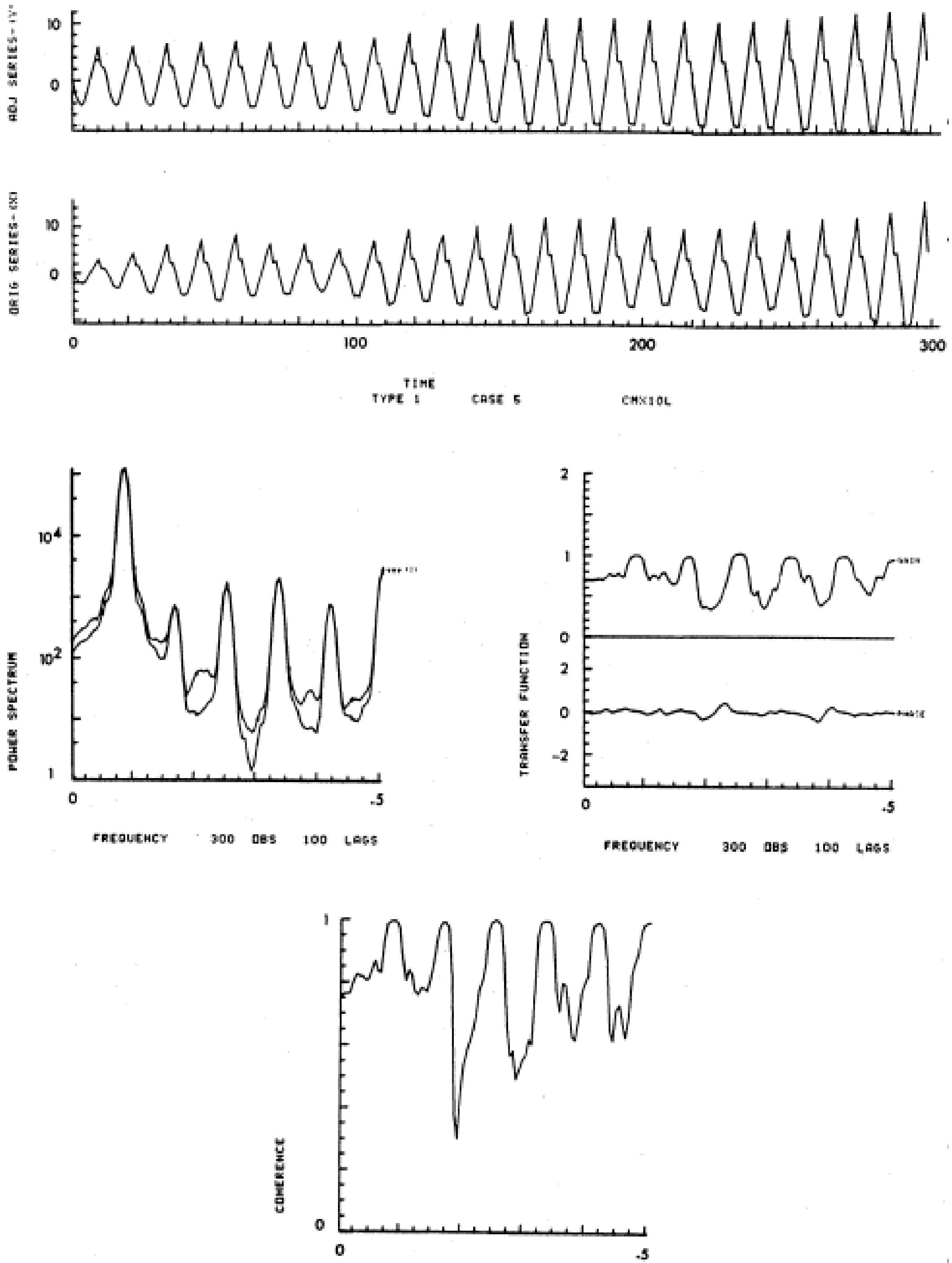
100
0
100
200
300
TIME
TYPE 1
CASE 5
CMX10L
POWER SPECTRUM
10^4
10^2
1
0
.5
FREQUENCY
300 OBS
100 LAGS
TRANSFER FUNCTION
2
1
0
2
0
−2
0
.5
FREQUENCY
300 OBS
100 LAGS
COHERENCE
1
0
0
.5

Figure 6.05.1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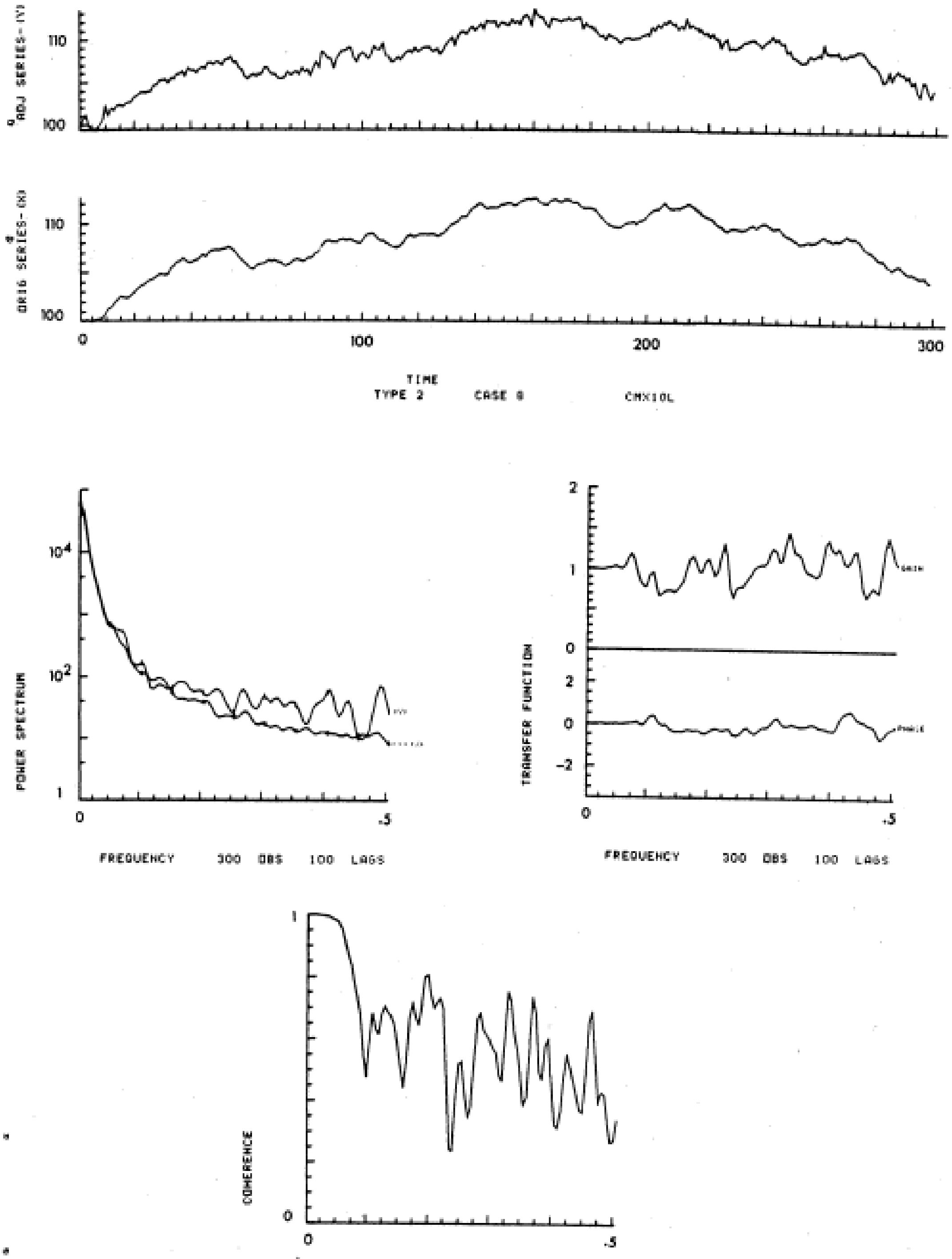
ORIG SERIES-(X)
TIME
TYPE 1
CASE 5
CMX10L
POWER SPECTRUM
FREQUENCY
300 OBS
100 LAGS
TRANSFER FUNCTION
FREQUENCY
300 OBS
100 LAGS
COHERENCE

Figure 6.05.2



ADJ SERIES-(Y)
ORIG SERIES-(X)
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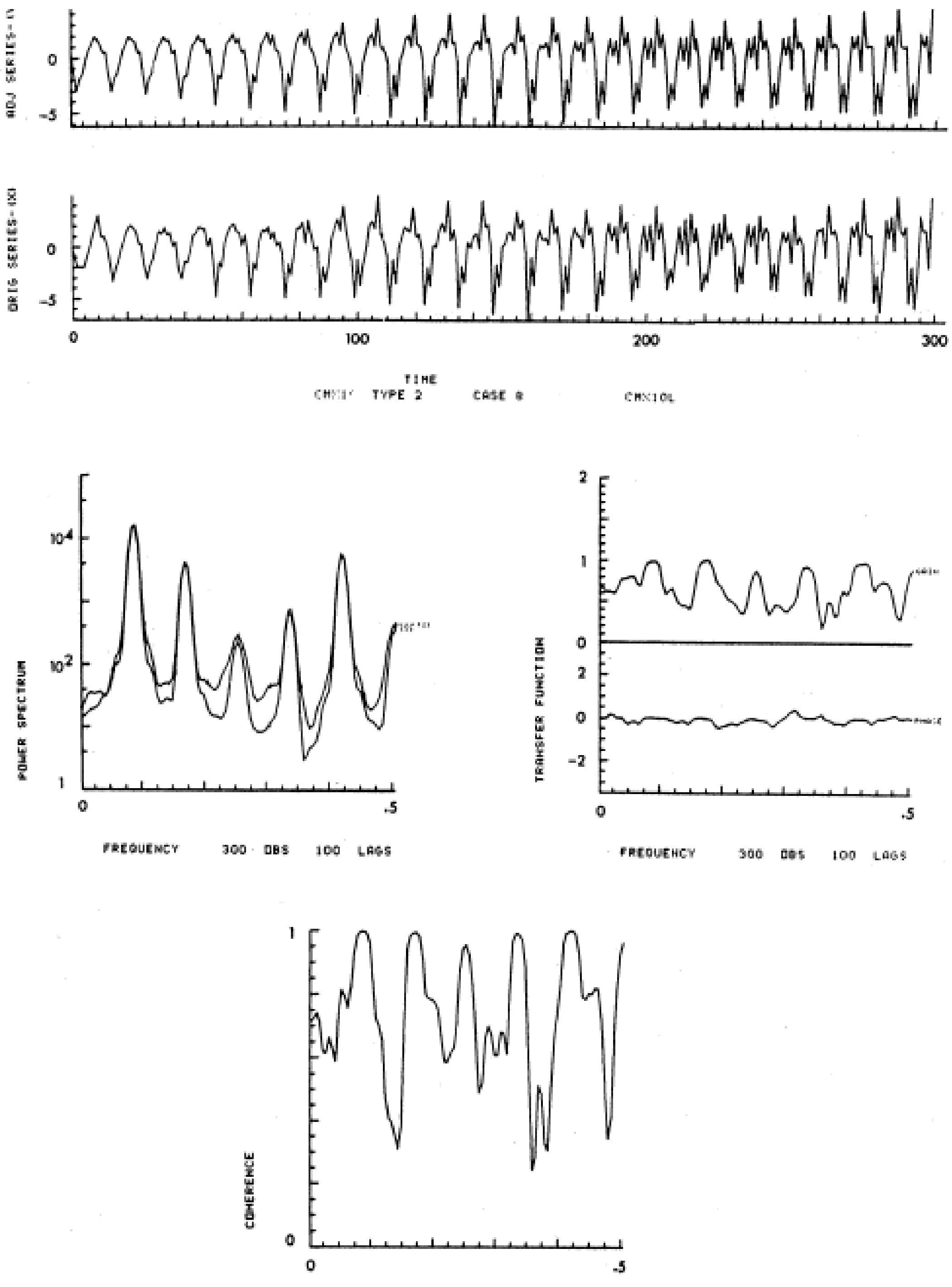
100
0
100
200
300
TIME
TYPE 2
POWER SPECTRUM
10^4
10^2
1
FREQUENCY
300 OBS
100 LAGS
TRANSFER FUNCTION
2
1
0
2
0
-2
.5
COHERENCE

Figure 6.06.1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ORIG SERIES
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0
100
200
300
TIME
TYPE 2
CASE 8
POWER SPECTRUM
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10^2
1
0
.5
FREQUENCY 300 OBS 100 LAGS
TRANSFER FUNCTION
2
1
0
2
0
-2
0
.5
FREQUENCY 300 OBS 100 LAGS
COHERENCE
1
0
0
.5

Figure 6.06.2

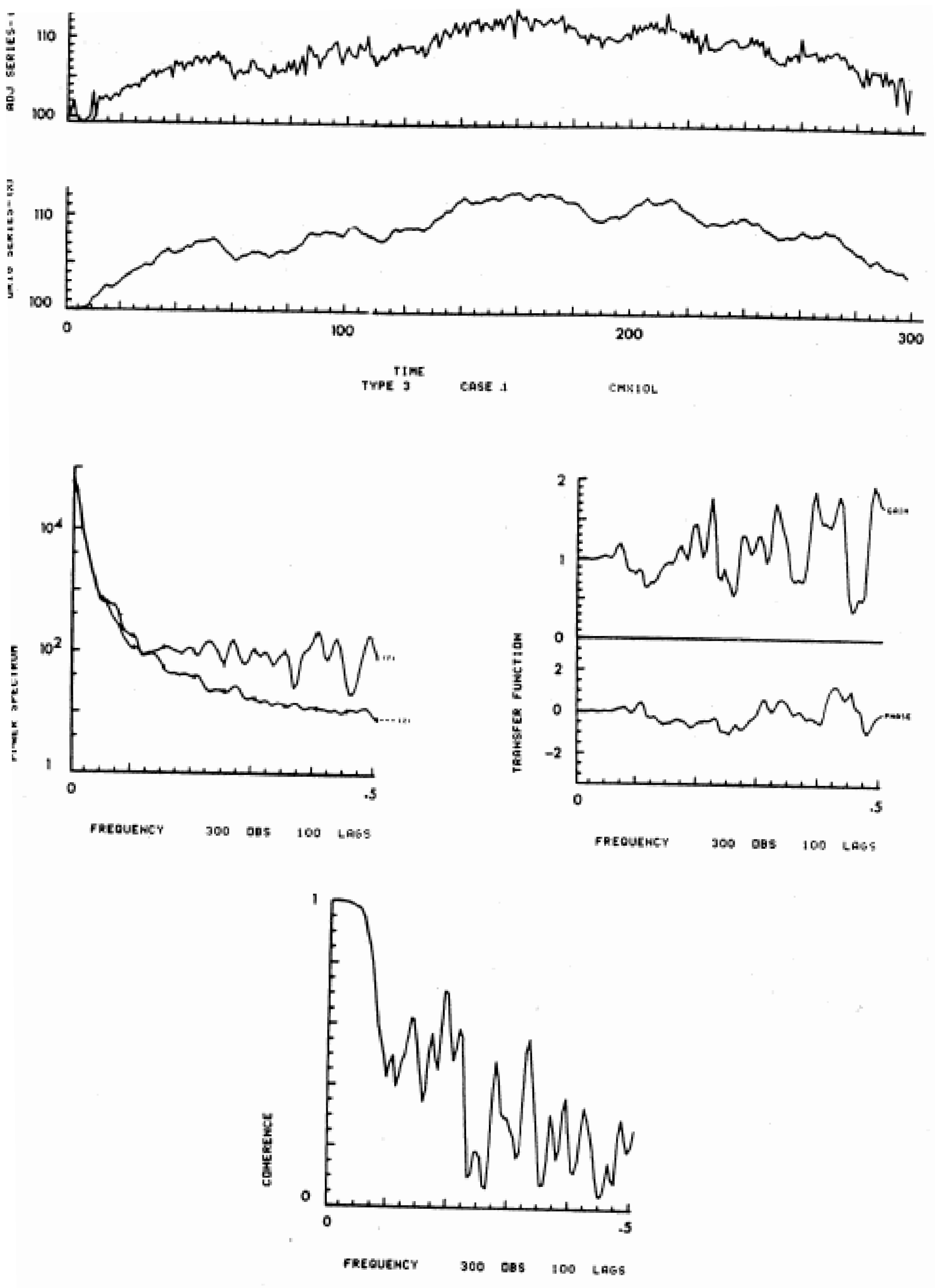
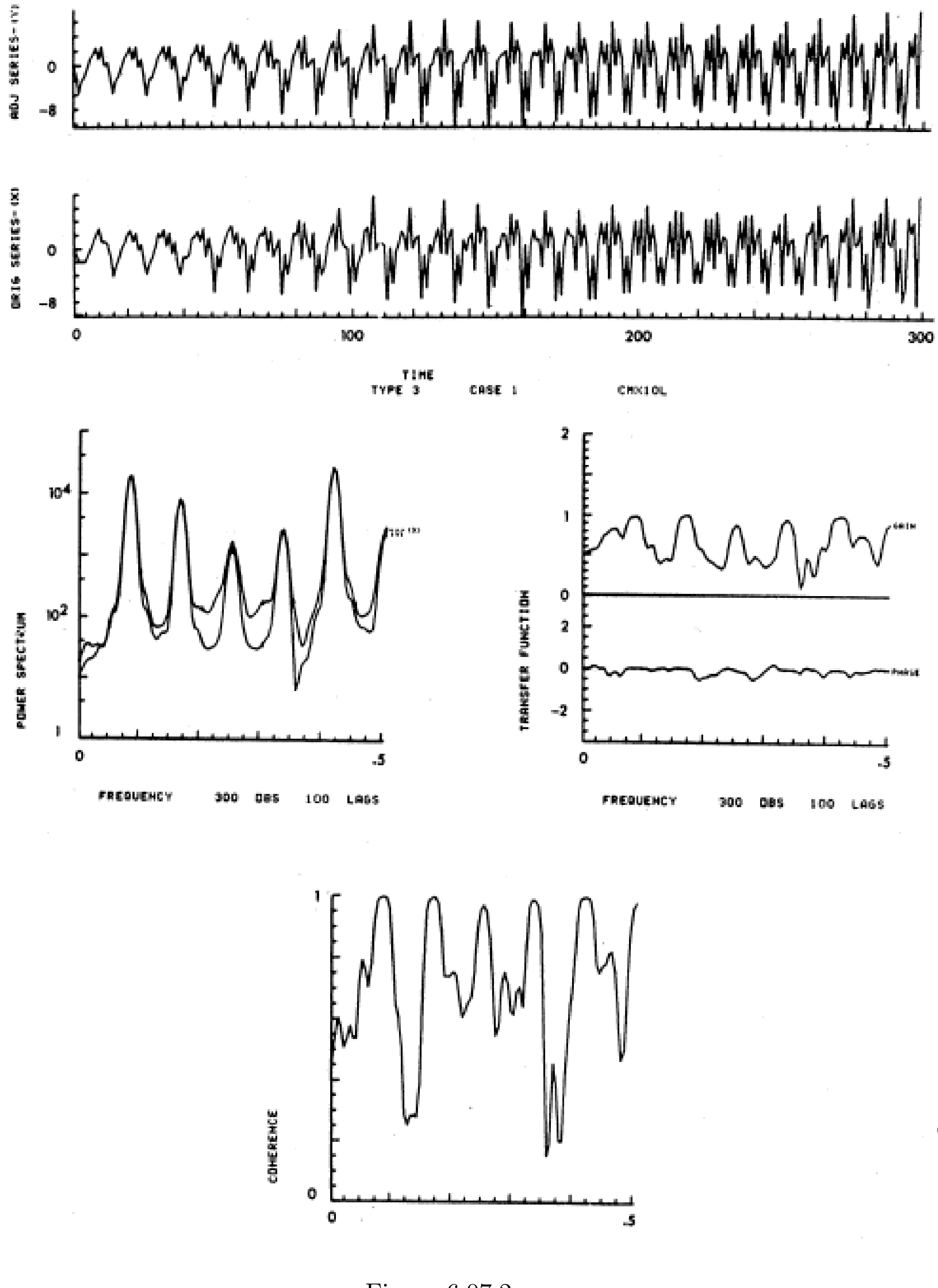

TIME
TYPE 3
CASE 1
FREQUENCY 300 OBS 100 LAGS
TRANSFER FUNCTION
COHERENCE

Figure 6.07.1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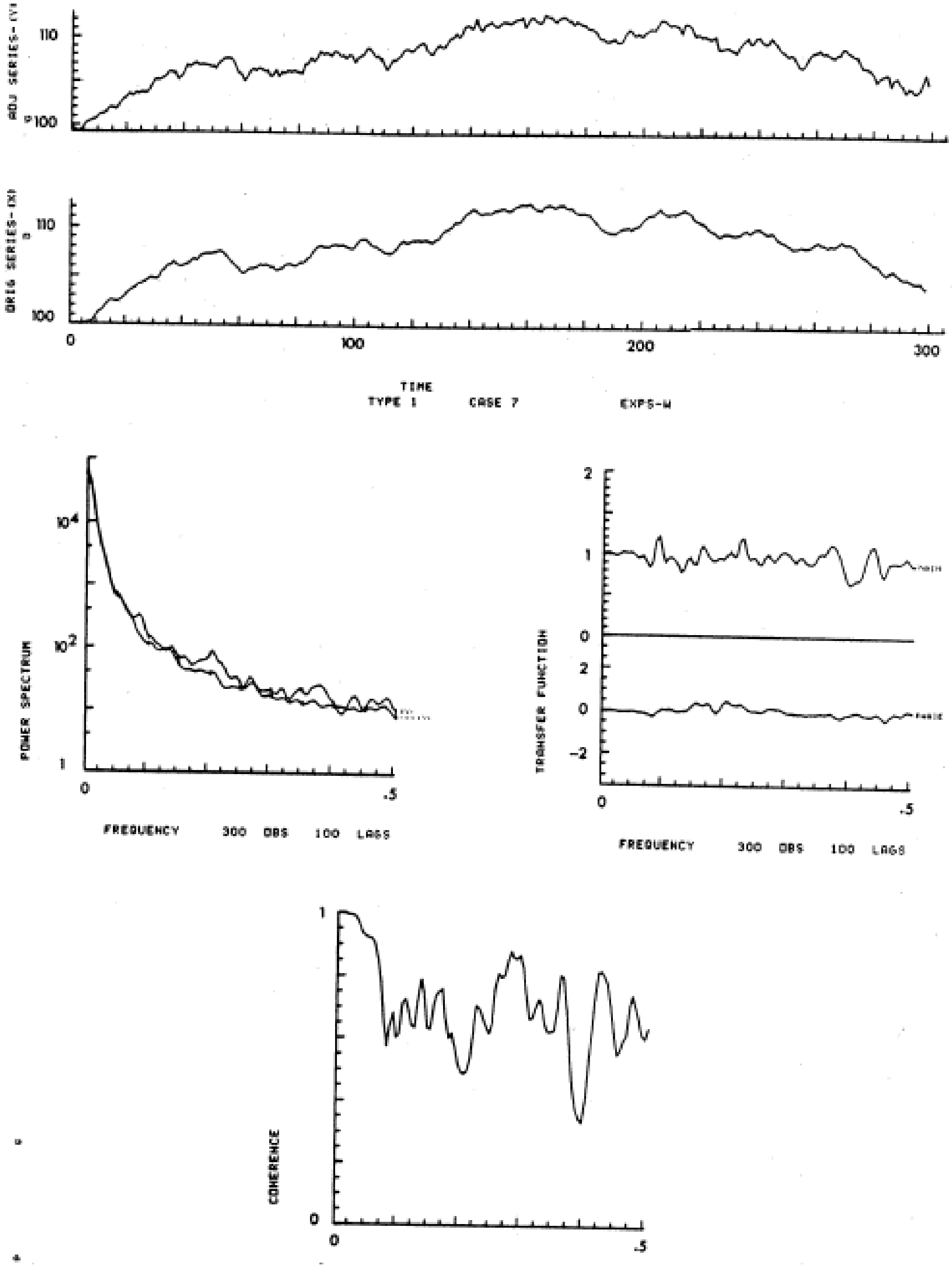
ORIG SERIES-(X)
0
-8
0
100
200
300
TIME
TYPE 3
CASE 1
CMX1OL
POWER SPECTRUM
10^4
10^2
1
0
.5
FREQUENCY
300 OBS
100 LAGS
TRANSFER FUNCTION
2
1
0
2
0
-2
0
.5
FREQUENCY
300 OBS
100 LAGS
COHERENCE
1
0
0
.5

Figure 6.07.2

6.4 THE RATIONAL FUNCTION METHOD

This method produces results that are quite naturally similar to the results obtained by Wald's method. While the method is in many ways similar to Wald's method, it represents an innovation in the use of a rational-function filter for the estimation of the monthly seasonal factors. This removes the need for the use of a long moving average of the observations for each month and also reduces the requirement of extrapolation of the final values at this stage. The rational-function filter which has been used requires only observations that are coincident with or precede the estimate in time. Thus the estimates for the last twelve months in the series are produced exactly as the previous ones. Therefore no correction of the estimates is needed as more data become available, except the correction required by the initial filtering of the data. In addition only the last three years of data and the initial values for the filter estimates are required for the estimation of the adjusted series for the current year. After the adjustment method has been applied to a series, then the final values of the rational-function filter estimates may be saved so that as new data become available, it is only necessary to use these estimated values and the last three years of data for the new estimation. The only figures that will be revised in the entire computation are those for the last 12 months of each estimation. These are the estimates which are based upon extrapolations of the 12-month moving average which was applied to the original observations.

The use of Wald's least-squares technique for estimation of the seasonal amplitude makes the performance of the rational-function method comparable to Wald's method when the seasonal pattern is constant but the amplitude is undergoing rapid change. Figure 6.08 shows the performance of the method under this condition. Figures 6.09 and 6.10 show the performance of the method when both the seasonal pattern and amplitude are changing.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100
ORIG SERIES-(X)
110
100
0
100
200
300
TIME
TYPE 1
CASE 7
EXPS-W
10⁴
10²
1
POWER SPECTRUM
0
.5
FREQUENCY
300 OBS
100 LAGS
2
1
0
2
0
-2
TRANSFER FUNCTION
0
.5
FREQUENCY
300 OBS
100 LAGS
1
0
COHERENCE
0
.5

Figure 6.08.1

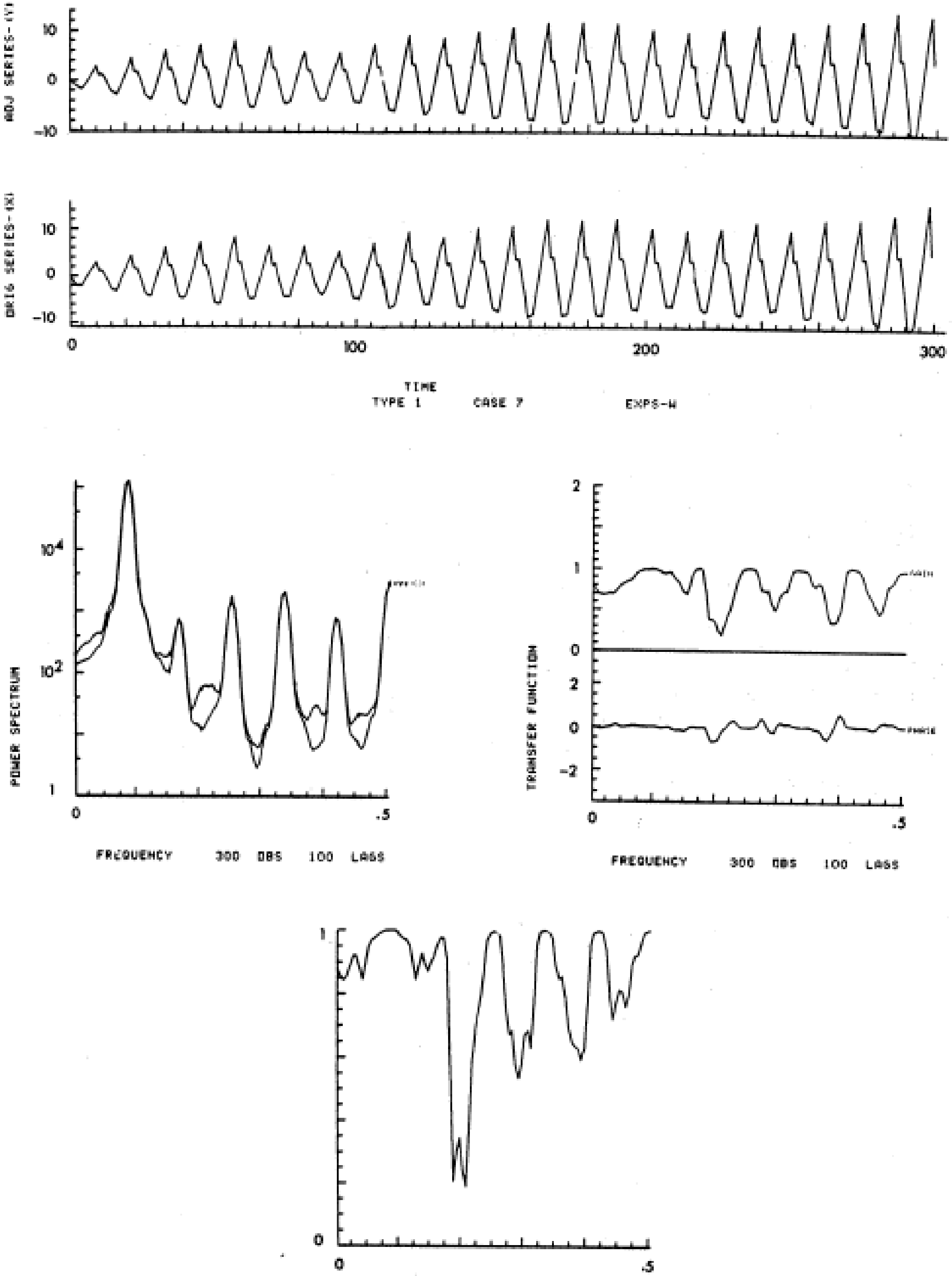
ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 1
CASE 7
EXPS-W
POWER SPECTRUM
FREQUENCY
300 OBS
100 LAGS
TRANSFER FUNCTION
FREQUENCY
300 OBS
100 LAGS

Figure 6.08.2

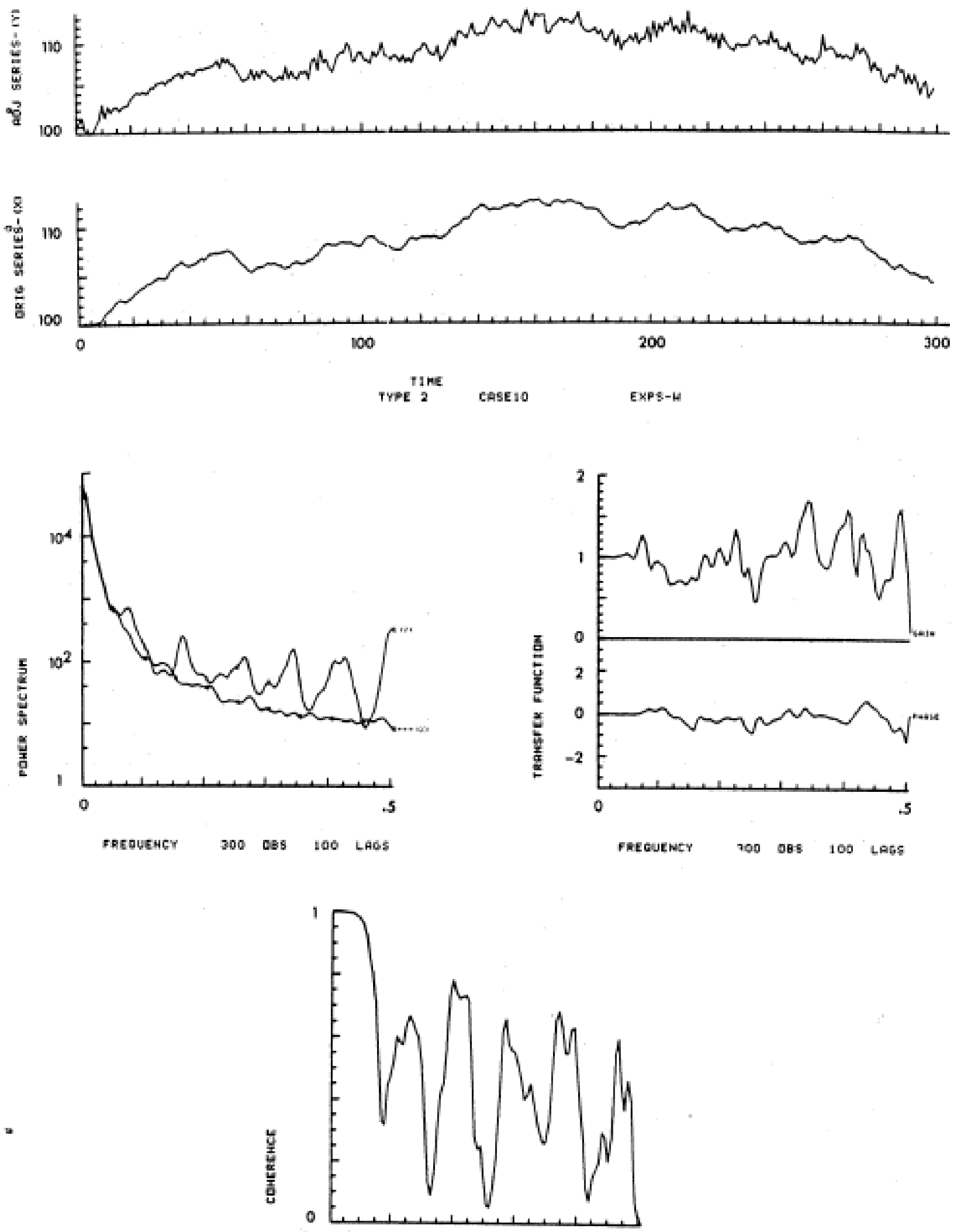
ADJ SERIES-(Y)
110
100
ORIG SERIES-(X)
110
100
0
100
200
300
TIME
TYPE 2
CASE10
EXPS-W
POWER SPECTRUM
10^4
10^2
1
0
.5
FREQUENCY 300 OBS 100 LAGS
TRANSFER FUNCTION
2
1
0
2
0
-2
0
.5
FREQUENCY 300 OBS 100 LAGS
COHERENCE
1
0
0
.5

Figure 6.09.1

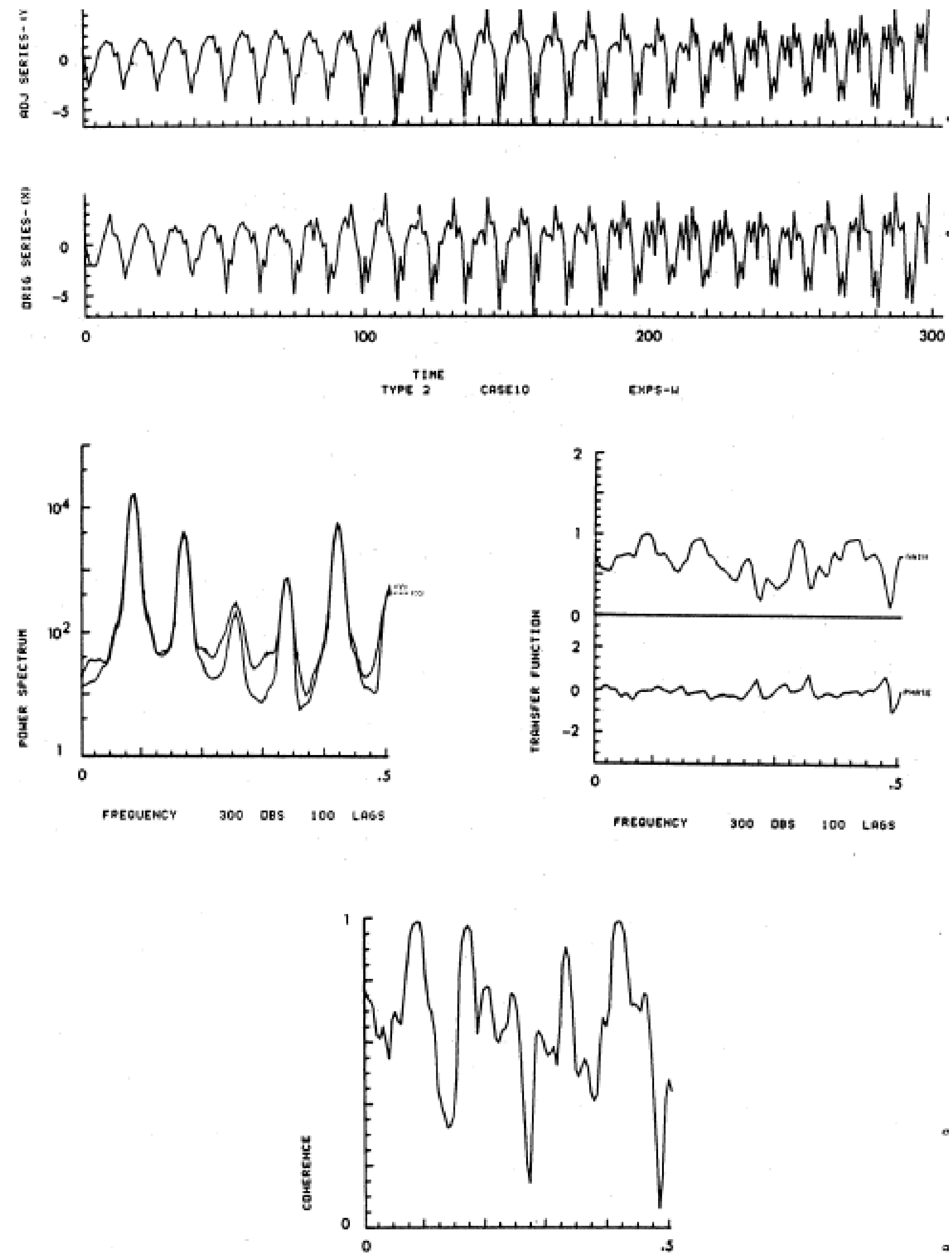
ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 2
CASE10
EXPS-W
POWER SPECTRUM
FREQUENCY 300 OBS 100 LAGS
TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY 300 OBS 100 LAGS
COHERENCE

Figure 6.09.2

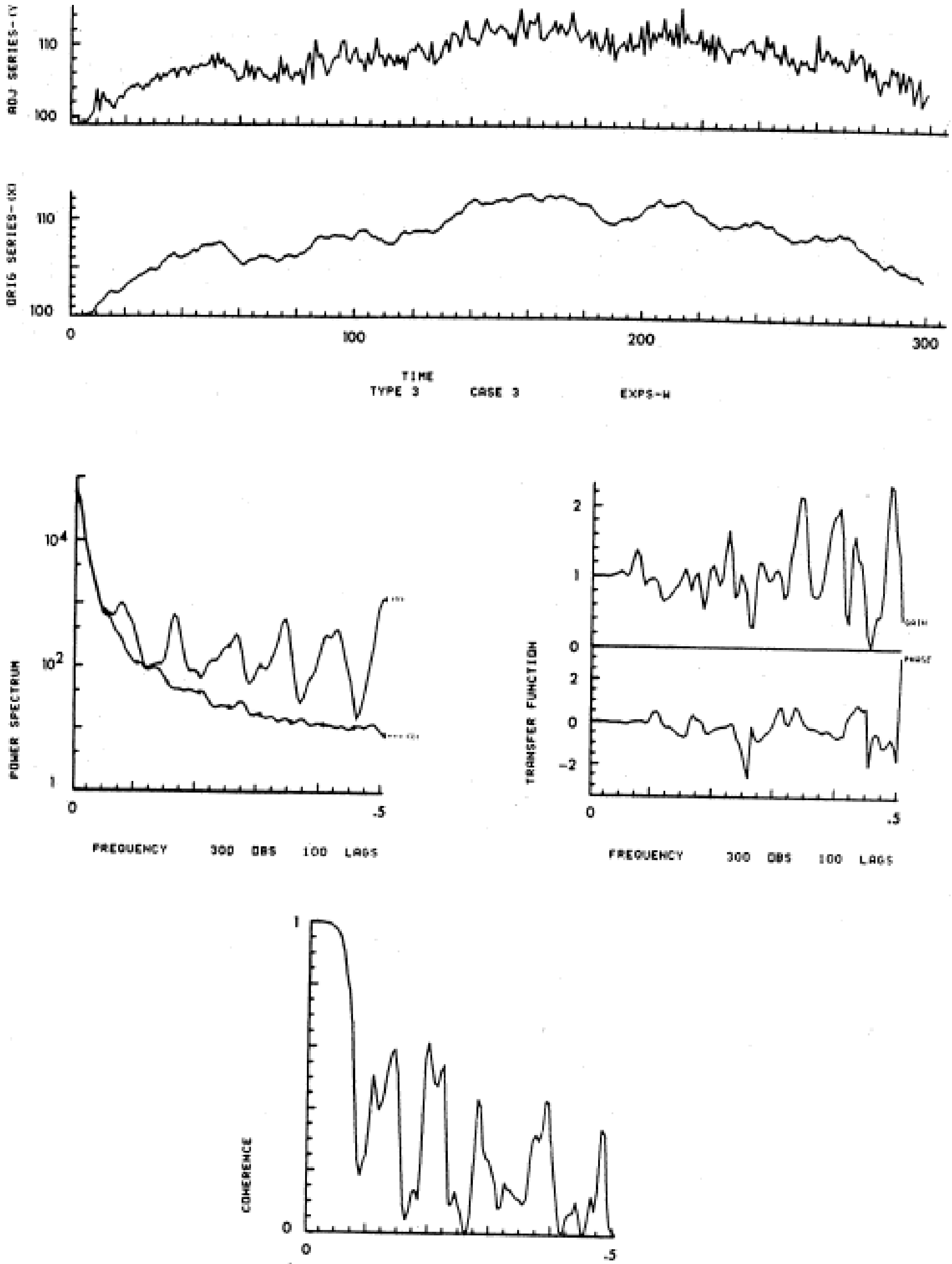
ADJ SERIES-(Y)
110
100
ORIG SERIES-(X)
110
100
0
100
200
300
TIME
TYPE 3
CASE 3
EXPS-W
POWER SPECTRUM
10^4
10^2
1
0
.5
FREQUENCY
300 OBS
100 LAGS
TRANSFER FUNCTION
2
1
0
GAIN
PHASE
2
0
-2
0
.5
FREQUENCY
300 OBS
100 LAGS
COHERENCE
1
0
0
.5

Figure 6.10.1

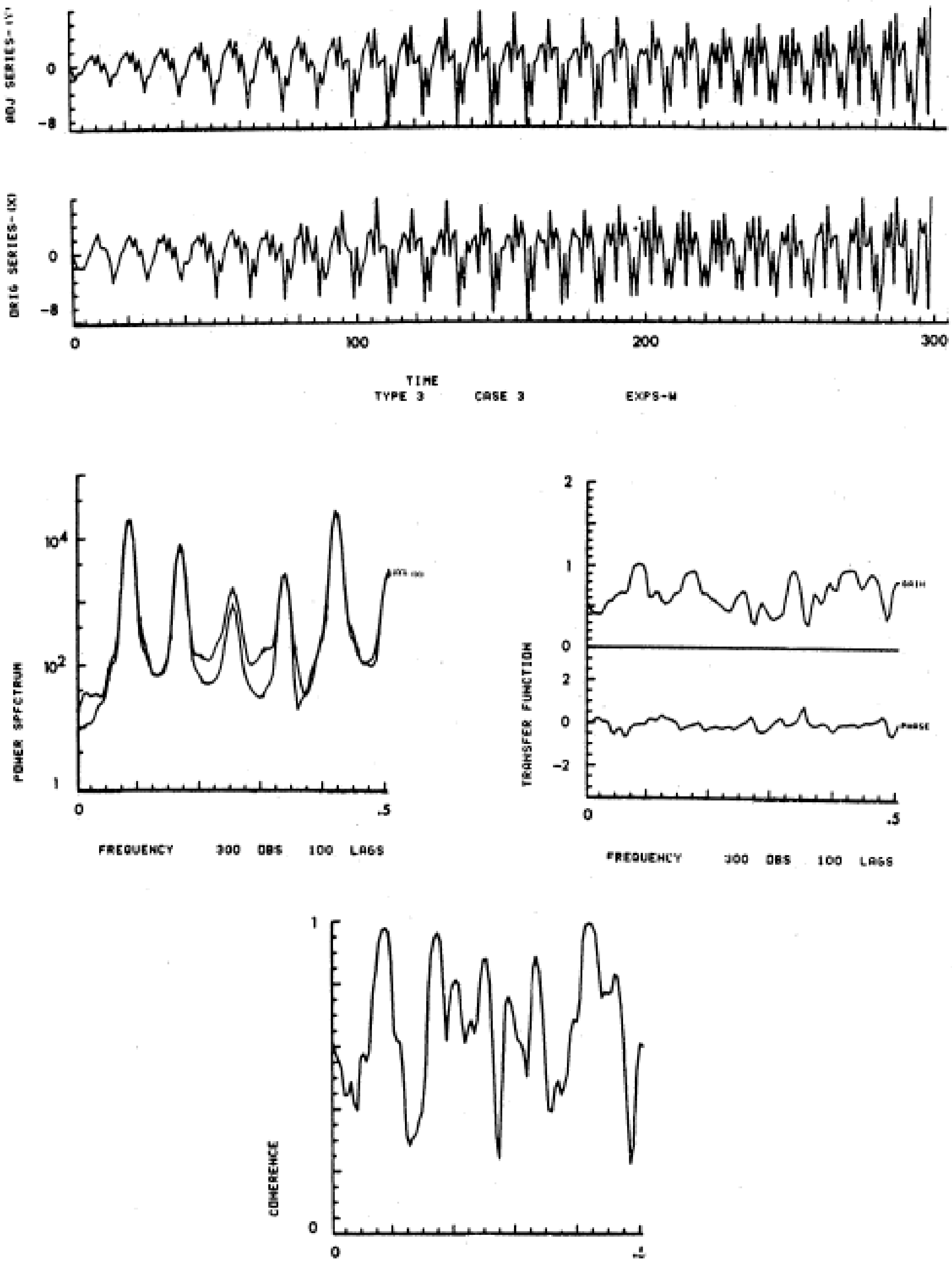
ADJ SERIES-(Y)
0
-8
ORIG SERIES-(X)
0
-8
0
100
200
300
TIME
TYPE 3
CASE 3
EXPS-W
10^4
10^2
1
0
.5
FREQUENCY
300 OBS
100 LAGS
2
1
0
2
0
-2
TRANSFER FUNCTION
0
.5
FREQUENCY
300 OBS
100 LAGS
1
COHERENCE
0
0

Figure 6.10.2

7. CONCLUSIONS

On the basis of the criteria put forward in the Introduction to this paper and the computations presented in Section 6.0, we have been able to compare the relative performances of various seasonal adjustment methods. The most important conclusions to be drawn from this analysis are as follows:

7.1 For the case of a constant, strictly periodic seasonal, it is difficult to see how it would be possible to improve upon Hannan's method. In addition, the use of transfer function analysis as applied in Hannan's method is of general applicability to seasonal adjustment methods which use moving averages.

7.2 Wald's method, which is unique in its use of the intercorrelation of the monthly seasonal factors, definitely displays the value of the use of this intercorrelation when the assumption of a constant seasonal pattern is met. When the amplitude (but not the pattern) of the seasonal varies relatively rapidly, Wald's method produces the best estimate of the underlying autoregressive series.

7.3 The value of the Census Method is that it fulfills the stated objectives reasonably well under all conditions. It continues to provide an adjusted series which closely approximates the autoregressive series even when both the amplitude and the pattern of the seasonal vary considerably. This result is not at all surprising given that the method uses a moving average estimate of the change in the seasonal and that it does not rely in any way upon intercorrelations between the monthly seasonal factors. However, there are two points at which the Census Method is relatively weak. The first is that no correction is made for the bias introduced by the use of the Spencer 15-point formula. As mentioned previously, this correction could easily be made in the Census Method without any change in the basic method. Therefore the presence of this bias in the current version of the method is not a fundamental criticism. The other weakness of the Census Method is that the frequency response of the moving average used to estimate the moving monthly seasonal factors is very low even at quite low frequencies. This characteristic produces a very stable estimate of the seasonal factors but a poor estimate of relatively rapid variations in the amplitude or pattern of the seasonal.

7.4 The rational-function method — basically an extension of Wald's and Hannan's methods — is an attempt, shown to be at least partially successful, to remove the two weaknesses of the Census Method mentioned above, while maintaining the generality of the assumptions on which the Census Method is based. The method is very simple both conceptually and computationally. In addition some of the problems of treatment of the end values of the series are avoided through the use of an asymmetric, single-sided, filter function. This method is not intended as a complete new method of seasonal adjustment, but is simply presented to show the potential value of the application of simple ideas of frequency decomposition to seasonal adjustment.

Finally, mention must be made of the cost involved in the use of a seasonal adjustment method which estimates relatively rapid changes in the seasonal. It is not possible to estimate rapid changes in the seasonal without, in some way, affecting the information in the series at frequencies near to the seasonal. Thus the more "flexible" the seasonal estimator, the greater the disturbance to the series at non-seasonal frequencies. In attempting to estimate a changing seasonal, it would generally seem to be desirable to use a method that does not completely remove the seasonal in order to reduce the amount of distortion at

non-seasonal frequencies. The exact trade-off between removal of a varying seasonal and distortion of the series is something which needs ultimately to be determined in the context of the intended use of the information in the series.

APPENDIX A: DERIVATION OF WALD'S METHOD

This appendix presents the derivation of equation (4.2.7), which defines the estimated seasonal coefficients. We continue the notation established in Section 4.2 and presuppose the statement of the assumptions given in that section.

The values of $p(i,k)$ for the kth month are the same for all i since $p(t)$ is a periodic function. Replacing these values by a common value $p(k)$, we obtain from equation (4.2.6):

$$\frac{1}{m}\sum_{i=1}^{m}\psi(i,k) \approx \frac{1}{m}p(k)\sum_{i=1}^{m}\lambda(i,k) + \frac{1}{m}\sum_{i=1}^{m}y(i,k) \qquad \text{for } k = 1, 2, \ldots, 12. \tag{A.1}$$

Now let

$$\frac{1}{m}\sum_{i=1}^{m}\lambda(i,k) = \lambda(k) \qquad \text{for } k = 1, 2, \ldots, 12 \tag{A.2}$$

Also, let

$$\frac{1}{12}\sum_{k=1}^{12}\lambda(k) = \lambda(0).$$

Then substituting in equation (A.2):

$$\frac{1}{12m}\sum_{k=1}^{12}\sum_{i=1}^{m}\lambda(i,k) = \lambda(0). \tag{A.3}$$

Replacing the $A(i,k)$ in (A.2) by $\frac{1}{12}\sum_{k=1}^{12}\lambda(i,k) + \delta(i,k)$, where $\delta(i,k)$ is the deviation of $\lambda(i,k)$ from its mean, results in:

$$\frac{1}{m}\sum_{i=1}^{m}\left[\frac{\sum_{k=1}^{12}\lambda(i,k)}{12}\right] = \lambda(k)$$

or

$$\frac{1}{12m}\sum_{i=1}^{m}\sum_{k=1}^{12}\lambda(i,k) + \frac{1}{m}\sum_{i=1}^{m}\delta(i,k) = \lambda(k)$$

Using (A.3) this becomes

$$\lambda(0) - \lambda(k) = -\frac{1}{m}\sum_{i=1}^{m}\delta(i,k). \tag{A.4}$$

Let us assume that the maximum deviation of $\lambda(t)$ within a year is v% of the mean of that year:

$$|\delta(i,k)| \leq \frac{v}{100 \times 12}\sum_{k=1}^{12}\lambda(i,k). \tag{A.5}$$

Taking the absolute values of the left and right sides of (A.4) one finds:

$$|\lambda(0) - \lambda(k)| = \left|\frac{1}{m}\sum_{i=1}^{m}\delta(i,k)\right| \leq \frac{v}{100 \times 12m}\sum_{i=1}^{m}\sum_{k=1}^{12}\lambda(i,k) = \frac{v}{100}\lambda(0). \tag{A.6}$$

In general, the $\delta(i,k)$ will have alternating signs and $\frac{1}{m}|\sum_{i=1}^{m}\delta(i,k)|$ will therefore be considerably smaller than $\frac{v}{100}\lambda(0)$. This justifies the assumption that:

$$\lambda(k) \approx \lambda(0) \qquad \text{for} \quad k = 1, 2, \ldots, 12. \tag{A.7}$$

Consequently, one can write for (A.1):

$$\frac{1}{m}\sum_{i=1}^{m}\psi(i,k) \approx p(k)\lambda(0) + \frac{1}{m}\sum_{i=1}^{m} y(i,k). \tag{A.8}$$

With respect to the residuals $z(t)$ the following two assumptions are made:

$$\frac{1}{12m}\sum_{i=1}^{m}\sum_{k=1}^{12} z(i,k) \approx 0$$

and further

$$\frac{1}{m}\sum_{i=1}^{m} z(i,k) \approx 0. \tag{A.9}$$

By definition $y(i,k) = z(i,k) - z^*(i,k)$. Therefore:

$$\sum_{i=1}^{m} y(i,k) = \sum_{i=1}^{m} z(i,k) - \sum_{i=1}^{m} z^*(i,k). \tag{A.10}$$

Since

$$\sum_{i=1}^{m} z^*(i,k) = \frac{1}{12}\sum_{k=1}^{12}\sum_{i=1}^{m} z(i,k),$$

equation (A.10), after dividing by m, becomes:

$$\frac{1}{m}\sum_{i=1}^{m} y(i,k) = \frac{1}{m}\sum_{i=1}^{m} z(i,k) - \frac{1}{12m}\sum_{i=1}^{m}\sum_{k=1}^{12} z(i,k).$$

From (A.9):

$$\frac{1}{m}\sum_{i=1}^{m} y(i,k) \approx 0.$$

Hence, (A.8) can be replaced by:

$$\frac{1}{m}\sum_{i=1}^{m}\psi(i,k) \approx \lambda(0)p(k) \quad \text{for} \quad k = 1, 2, \ldots, 12. \tag{A.11}$$

The 12 sums, one for each month, on the left side of (A.11) can be obtained from the $\psi(i,k)$ series. What is left to be done is to split these sums into the two components $\lambda(0)$ and $p(k)$, where $\lambda(0)$ is a constant, being the arithmetic mean of all $\lambda(i,k)$. The $\lambda(i,k)$ and $p(k)$ are, however, still to be determined. From (4.2.5), we can write, after replacing $s(t)$ by $\lambda(t)p(t)$:

$$\psi(i,k) \approx \lambda(i,k)p(i,k) + y(i,k) \quad \text{for} \quad i = 1, 2, \ldots, m$$
$$k = 1, 2, \ldots, 12,$$

or, since the function $p(t)$ is periodic:

$$\psi(i,k) \approx \lambda(i,k)p(k) + y(i,k).$$

Then

$$\psi(i,k) \approx \frac{\lambda(i,k)}{\lambda(0)}\lambda(0)p(k) + y(i,k),$$

or, from (A.11)

$$\psi(,k) \approx \frac{\lambda(i,k)}{\psi(0)}\frac{1}{m}\sum_{i=1}^{m}\psi(i,k) + y(i,k)$$

and

$$\psi(i,k) \approx \mu(i,k)a(k) + y(i,k) \tag{A.12}$$

where

$$\mu(i,k) \approx \frac{\lambda(i,k)}{\lambda(0)} \quad \text{and} \quad a(k) = \frac{1}{m}\sum_{i=1}^{m}\psi(i,k).$$

Now the $\mu(i,k)$ remain to be determined.

(A.12) may also be written:

$$y(i,k) \approx \psi(i,k) - \mu(i,k)a(k) \quad \text{for} \quad i = 1,2,\ldots,m$$
$$k = 1,2,\ldots,12. \tag{A.13}$$

Since $y(t)$ may be assumed to be a normally distributed random variable, the $\mu(t,k)$ may be determined so as to minimize:

$$\sum_{i=1}^{m}\sum_{k=1}^{12}[\psi(i,k) - \mu(i,k)a(k)]^2.$$

The additional condition to be imposed on the function $\mu(t) = \frac{\lambda(0)}{\lambda(t)}$ is that its value will change only slowly over time. This leads to the assumption that the $\mu(i,k)$ may be approximated by minimizing:

$$\sum_{j=k-6}^{k+5}[\psi(i,j) - \mu(i,k)a(j)]^2 \quad k = 1,2,\ldots,12 \tag{A.14}$$

subject to the condition that for each point t in time, the value of $\mu(t)$ will be nearly constant in each period $(t-6, t+5)$ for $t = 7,8,\ldots,(n-5)$.[1]

Rather than minimizing the expression $[\psi(i,k) - \mu(i,k)a(k)]^2$ with respect to $\mu(i,k)$ over the whole period $t = 7,8,\ldots,(n-5)$, it is minimized over consecutive 12-month periods. The first period includes the time points $7,8,\ldots,18$, the second one $8,9,\ldots,19$ etc., the last one $(n-16),(n-15),\ldots,(n-5)$; there are, therefore, $n-6$ periods all together. The condition imposed on the $\mu(t)$ assures that the 12 μ's related to the first period, $\mu(1),\mu(2),\ldots,\mu(12)$ will all have the same value, say $C(1)$. Similarly, the 12 μ's related to the jth period, $\mu(j),\mu(j+1),\ldots,\mu(j+11)$, will have the same value $C(j)$, where, in general $C(1) \neq C(2) \neq$

[1] It should be noted that if in (A.14) the index j of $\psi(i,j)$ and $a(j)$ becomes ≤ 0, the former are replaced by $\psi(i-1,j+12)$ and $a(j+12)$ and if $j \geq 12$, by $\psi(i+1,j-12)$ and $a(j-12)$.

$\ldots \neq C(j)$. A particular μ, say $\mu(i)$, is determined when the minimization of equation (A.14) has been performed for the 12-month period i.

Replacing the 12 $\mu(i,k)$'s in (A.14) by a single $\mu(i,k)$ and differentiating that expression with respect to that $\mu(i,k)$ we have:

$$\mu(i,k) = \frac{\sum_{j=k-6}^{k+5} \psi(i,j)a(j)}{\sum_{j=k-6}^{k+5} [a(j)]^2}. \tag{A.15}$$

The seasonal fluctuations $s(i,k)$ can now directly be derived from these $\mu(i,k)$'s:

$$s(i,k) = a(k)\mu(i,k) = a(k)\frac{\sum_{j=k-6}^{k+5} \psi(i,j)a(j)}{\sum_{j=k-6}^{k+5} [a(j)]^2}. \tag{A.16}$$

Since the same 12 $a(k)$'s appear in the denominator repeatedly, formula (A.16) can be still further simplified to:

$$s(i,k) = a(k)\frac{\sum_{j=k-6}^{k+5} \psi(i,j)a(j)}{\sum_{l=1}^{12} [a(l)]^2}. \tag{A.17}$$

Now, the $s(i,k)$ can be obtained from the original observations $\varphi(t)$ via the series of differences $\psi(t)$ and the $a(k)$ derived from the $\psi(t)$.

Wald also includes in his monograph the following scheme for carrying out the computational steps:

1. First $\varphi^*(t)$, the 12-month moving average of $\varphi(t)$, is computed.
2. Then the differences $\psi(t) = \varphi(t) - \varphi^*(t)$ are formed and put in the form of a matrix with 12 columns, one for each month. At this point, it should be remarked that if in actual situations the values of some of the $\psi(i,k)$ are too extreme, due to special circumstances (strikes, for example), then these $\psi(i,k)$ may be excluded from the rest of the computations.
3. For each month $k (k = 1, 2, \ldots, 12)$, the arithmetic mean $a(k)$ of the values of $\psi(i,k)$ which appear in the kth column of the matrix are then computed.
4. This is followed by an adjustment of the $a(k)$ according to the formula:

$$a'(k) = a(k) - |a(k)|\frac{a(1) + a(2) + \ldots + a(12)}{|a(1)| + |a(2)| + \ldots + |a(12)|} \tag{A.18}$$

so that

$$\sum_{k=1}^{12} a'(k) = 0.$$

5. Then the series $F(i,k) = b'(k)\psi(i,k)$ is formed where

$$b'(k) = \frac{a'(k)}{\sum_{l=1}^{12} [a'(l)]^2}.$$

6. Adding the first 12 values of the series $F(i,k)$ will give $\mu(1)$, the 1st term of the $\mu(t)$ series. Subtracting from $\mu(1)$ the 1st term of $F(i,k)$ and adding to it the 13th term

of $F(i,k)$ will give $\mu(2)$, the 2nd term of the $\mu(t)$ series. This procedure is continued until the last term of $F(i,k)$ has been added and the (last -12)th term of $F(i,k)$ has been subtracted from $\mu(t-1)$ to give $\mu(t)$. After the $\mu(t)$ series has been computed, it is arranged in the form of a matrix with 12 columns, one column for each month.

7. The seasonal fluctuations $s(i,k)$ are then computed according to the formula:

$$s(i,k) = a'(k)\mu(i,k). \tag{A.19}$$

   It is not possible to obtain values for $p(t)$ and $s(t)$ for the last 11 months this way, which is a serious drawback in practical applications. Wald suggests in (p. 427[27]) that the simplest solution to this problem is to add 11 more terms to the $\mu(t)$ series, all equal to the last computed $\mu(t)$. This will then make it possible to compute the $s(t)$ for the last 11 months on the basis of (A.19). However, the extrapolation technique described in Section 3.0 is shown by Wald to be a better procedure.

8. Finally the seasonal series $s(t)$ is subtracted from the series $\varphi(t)$ to give the seasonally adjusted series.

APPENDIX B: DETAILS OF COMPUTATIONS

While analysis using the computation of spectra and cross-spectra is becoming more widespread in economics, the technique is neither completely accessible to all mathematical economists nor is it fully standardized. Therefore the following definitions and equations will be stated.

The serial cross-correlation coefficient for two series, x_t and y_t, (which may be identical) is defined by:

$$R_{xy}(s) = \frac{\sum_{t=1}^{N-s}(x_t-\bar{x})(y_{t+s}-\bar{y})}{\Big[\sum_{t=1}^{N}(x_t-\bar{x})^2\sum_{t=1}^{N}(y_t-\bar{y})^2\Big]^{1/2}} \qquad s = 0.\ldots,m$$

where: N – number of observations, and

$$\bar{x} = \frac{1}{N}\sum_{t=1}^{N} x_t$$

$$\bar{y} = \frac{1}{N}\sum_{t=1}^{N} y_t$$

$$R_{xy}(s) = R_{yx}(-s)$$

$$R_{xx}(s) = R_{xx}(-s)$$

The Parzen estimate of the normalized spectrum is then given by:

$$F_{xy}(\omega) = \sum_{s=-m}^{m} R_{xy}(s)\lambda(s)e^{i\omega s}$$

where:

$$\begin{aligned}\lambda(s) &= 1-6\Big(\frac{s}{m}\Big)^2+6\Big(\Big|\frac{s}{m}\Big|\Big)^3 && s \le \frac{m}{2}\\ &= 2\Big(1-\Big|\frac{s}{m}\Big|\Big)^3 && s > \frac{m}{2}.\end{aligned}$$

Due to

$$R_{xx}(s) = R_{xx}(-s)$$

$$F_{xx}(\omega) = R_{xx}(0) + 2\sum_{s=1}^{m} R_{xx}(s)\lambda(s)\cos\omega s.$$

Writing $F_{xy}(\omega)$ in terms of its real and imaginary parts we have the real part, the co-spectrum:

$$C_{xy}(\omega) = R_{xy}(0) + \sum_{s=1}^{m}[R_{xy}(s) + R_{yx}(s)]\lambda(s)\cos\omega s$$

and the imaginary part, the quadrature spectrum:

$$Q_{xy}(\omega) = \sum_{s=1}^{m}[R_{xy}(s) - R_{yx}(s)]\lambda(s)\sin\omega s$$

From these we define the following estimates:

1. Coherence

$$S_{xy}^2(\omega) = \frac{|F_{xy}(\omega)|^2}{F_{xx}(\omega)F_{yy}(\omega)}$$

2. Gain

$$G_{xy}(\omega) = \frac{|F_{xy}(\omega)|}{F_{xx}(\omega)}$$

3. Phase

$$\Phi_{xy}(\omega) = \tan^{-1}\frac{Q_{xy}(\omega)}{C_{xy}(\omega)}.$$

All of the series computed for this paper were made up of 300 observations. One hundred lags were used. Thus $N = 300, m = 100$.

All of the computations were Fortran programmed for the Princeton University IBM 7090. Extensive use was made of the on-line graphic display and recording facilities on the computer. In fact only summary statistics and identifying comments were output on tape for subsequent printing. All other results, of which the figures in the paper are examples, were recorded directly on 35 mm microfilm. Without this graphic output facility, the development of the various programs would have been much slower, the analysis of results vastly slower and more laborious, and the analysis and development of the rational-function method nearly impossible within the time available.

Appendix C: Fortran Program Information and Listing

APPENDIX C: FORTRAN PROGRAM INFORMATION AND LISTINGS

The programs listed below were used for the rational-function method described above. They were completed in 1964 and were used experimentally by the Bureau of Labor Statistics and to some extent by the Census Bureau. In the end, both groups decided to continue with the Census Method. In particular, Julius Shiskin took a considerable interest in this work. We had several long discussions about the methods described here. His, and the Bureau's, investment in and experience with the Census Method precluded a radical change.

The programs are all in Fortran. I see no useful purpose in converting them to another language other than, possibly, Octave. On any modern computing system running "make" should compile and create an executable for use as described below. There are three programs: season.f seaexps.f seafilt.f. These have been verified using the GNU Fortran which implements ISO/IEC 1539:1997 (Fortran 95).

These programs and some additional scripts and example data files are available in a "tar" file at: google drive folders/RM-64. Enter the RM-64 folder and download the tar file: seas-adjust-progs.tar. The tar file contains a makefile which is written to do the compilation. Just type "make." This should produce the executable program: season. The current season executable was created using Fedora.

The data are read from fort.8 and the input and output time series (in Octave/Matlab format) are written to fort.9 and fort.10.

```
Making the links:
ln -s sourcedata.dat fort.8
ln -s fort.9 inputs.m
ln -s fort.10 outputs.m
```

makes it easy to just run season, then read the results into Octave or other programs for display or analysis of the results.

There are 2 example time series. The first uses: `Airdata_1946_60.dat`
`Typing:`

```
ln -s Airdata_1946_60.dat fort.8

season

octave --persist plot_data.m
```

is intended to compute the seasonally adjusted series from the Airline Passenger data and plot the original and adjusted series. The other example uses: `UK_unempl_CSO_1949_72.dat` in place of the "Airdata" above.

The input data file `sourcedata.dat` should be in the format:

line 1: Data description (for title of plots, etc)

line 2: FORMAT N M

This line is read using the format: `(3A4, I4, F6.0)`

line 3: The data arranged according to the FORMAT declaration on following lines where:

FORMAT is a Fortran Format declaration (such as `(12F5.0)`) to be used to read the data.

N total number of data values.

M number of values per year (normally 12).

The Octave script datamtoseason.m can be used to convert Octave arrays into the above format. See the script below for details.

Appendix C: Fortran Program Information and Listing

Program Listings

Seasonal adjustment Main Program: SEASON

```
C     Programs used in:
C     A Spectrum Analysis of Seasonal Adjustment
C     MAIN PROGRAM FOR SEASONAL ADJUSTMENT. MD GODFREY.
C     by: Godfrey and Karreman,
C     RM 64 Econometric Research Program
C     Princeton University
C
      COMMON T(500), S(500)
C     REAL   COR(120,4),SD(101,2,1), HISO(27,2), XA(27,2), XN(27,2)
      REAL   SM(12), A(2), B(3), SER(12,50)
      CHARACTER*32 TITLE
      CHARACTER*12 FORM
      DATA   A, B, PHI, KF/1.57,-.7,.03,.04,.06,3.1415926,2/
      LOGICAL DUMS
      DATA DUMS/.TRUE./

 90   CONTINUE
      WRITE(6, 1002)
 1002 FORMAT('Provide data description:')
      READ(8, 1004, END = 500)TITLE
 1004 FORMAT(A72)
      WRITE(6,1005)TITLE
 1005 FORMAT('Data title: ',A32)
      WRITE(6, 1008)
 1008 FORMAT('READ DATA FORMAT')
      READ(8, 1010, END = 500)FORM, N, P4
 1010 FORMAT(A12, I4, F6.0)
      WRITE(6, 1012)FORM, N, P4
 1012 FORMAT('READ DATA:', A12, I4, F6.0)
      READ(8, FORM)(S(K),K = 1,N)
C     WRITE DATA DESCRIPTION
      WRITE(9, 1015) TITLE
 1015 FORMAT(15Hplot_title = ' , A32, 2H';)
      WRITE(9, 1013)
 1013 FORMAT('in  =  [')
      WRITE(9, 1020)(S(K),K = 1,N)
      WRITE(9, 1022)
      DE = PHI/P4
      NL = 6
      CALL EXPS(A, N, DE, NL, KF, B)
      DO 100 K = 1, N
      SER(K,1) = T(K)
 100  CONTINUE
      NM = (N+11)/12
      NMLAST = N-(NM-1)*12
      IF(DUMS)GOTO 300
      DO 120 L = 1, 12
      SM(L) = 0.
      DO 110 K = 1, NM
      SM(L) = SM(L)+SER(L,K)
 110  CONTINUE
      SM(L) = SM(L)/FLOAT(NM)
      DO 1201 K = 1, NM
      SER(L,K) = SER(L,K)/SM(L)
 1201 CONTINUE
 120  CONTINUE
 300  CONTINUE
      WRITE(6,1012)
C     WRITE RESULT VECTOR TO out.m AND TO OUTPUT STREAM
      WRITE(10, 1014)
 1014 FORMAT('out  =  [')
      DO 200 K = 1, NM-1
      WRITE(6, 1020)(SER(L,K), L = 1,12)
      WRITE(10,1020)(SER(L,K), L = 1,12)
 200  CONTINUE
      IF(NMLAST .EQ. 0) GOTO 220
```

```
      WRITE(6, 1020)(SER(L,NM), L = 1,NMLAST)
      WRITE(10,1020)(SER(L,NM), L = 1,NMLAST)
220   CONTINUE
1020  FORMAT(2X,12F5.0, ' ...')
      WRITE(10, 1022)
1022  FORMAT(' ];')
      GOTO 90
500   CONTINUE
      WRITE(6, 1030)
1030  FORMAT('END OF FILE REACHED.')
      CALL EXIT
      END
```

Seasonal Adjustment rational-function routine: SEAEXPS

```
C     RATIONAL (WALD) VERSION OF EXPS
C
      SUBROUTINE EXPS(A, NT, DE, NL, KF, B)
      COMMON T(500), S(500)
      DIMENSION SX(12,42), B(21), A(21), V(24)

      CALL FILT(NT, 0.0, DE, NL, V, KF)
      M = NT/12
      DO 2 K = 1, 12
      SX(K,1) = 0.0
      DO 1 J = 2,5
      L = 12*J+K-12
      SX(K,1) = SX(K,1)+T(L)
1     CONTINUE
      SX(K,1) = SX(K,1)/4.0
      SX(K,2) = SX(K,1)
2     CONTINUE
      M1 = M+1
      DO 3 K = 1,12
      LP = NT+K
      T(LP) = T(LP-12)
      DO 31 J = 3,M1
      L = 12*J+K
      SX(K,J) = A(1)*SX(K,J-1)+A(2)*SX(K,J-2)+B(1)*T(L)+B(2)*T(L-12)
     1 +B(3)*T(L-24)
31    CONTINUE
3     CONTINUE
      DO 8 J = 2,M
      DO 81 K = 1,12
      S1 = .0
      S2 = .0
      NT1 = K+6
      NT2 = K+17
      DO 7 L = NT1,NT2
      M2 = L+12*(J-2)
      S1 = S1+T(M2)*SX(L,J-1)
      S2 = S2+SX(L,J-1)**2
7     CONTINUE
      SX(K,J-1) = SX(K,J)*S1/S2
81    CONTINUE
8     CONTINUE
      DO 11 J = 1,M
      L = M1-J
      DO 111 K = 1,12
      SX(K,L+1) = SX(K,L)
111   CONTINUE
11    CONTINUE
      DO 9 K = 1,12
      SX(K,1) = 2.0*SX(K,2)-SX(K,3)
9     CONTINUE
      DO 6 J1 = 1,M
      DO 61 J = 1,12
      SX(J,J1) = 0.0
      DO 62 K = 6,17
      L = J+K-1
```

```
      SX(J,J1) = SX(J,J1)+V(K)*SX(L,J1)
 62   CONTINUE
 61   CONTINUE
 6    CONTINUE
      DO 5 J = 1,M
      L = M1-J
      DO 51 K = 1,12
       SX(K,L+1) = SX(K,L)
 51   CONTINUE
 5    CONTINUE
      DO 12 K = 1,12
      SX(K,1) = 2.0*SX(K,2)-SX(K,3)
 12   CONTINUE
      DO 4 K = 1,NT
      T(K) = S(K)-SX(K,1)
 4    CONTINUE
      WRITE (6,10)
 10   FORMAT(/40H THIS IS RATIONAL (WALD) VERSION OF EXPS /)
      RETURN
      END
```

Seasonal Adjustment filtering routine: SEAFILT

```
C     BAND FILTER
C
      SUBROUTINE FILT(NT, AF, DE, NL, V, KF)

      DIMENSION AS(12), WS(12), AT(5,2), V(24)
      COMMON  FD(500), DA(500)

      P = 3.1415927
      NL1 = NL+1
      NL2 = 2*NL
      DO 299 J = 1, NL1
      WS(J) = -1.0
 299  CONTINUE
      GOTO(1, 4),KF
 1    CONTINUE
      DO 2 J = 1, NL1
      WS(J) = 1.0/12.0
 2    CONTINUE
      WS(NL1) = 1./24.
      WRITE (6,3)
 3    FORMAT(' SIMPLE MOVING AVE.')
      GOTO14
 4    CONTINUE
      WRITE (6,5)AF,DE,NL2
 5    FORMAT(' CENTER FREQ: ',F8.4,
     . ' BAND WIDTH: ',F8.4,I4,' TERMS.')
      AQ = AF*P/10.0
      AM = NL
      DO 10 L = 1, 12
      AL = L-1
      AL = AL*P/11.0
      AS(L) = DE
      DO 9 J = 1,NL
      AJ = J
      AIF = AJ-AM/2.0
C      IF(AIF)6,7,7
      IF(AIF .LT. 0) GOTO 6
      IF(AIF .GE. 0) GOTO 7
C      IF(AJ-AM/2.0)6,7,7
 6    B = 1.0-6.0*(AJ/AM)**2+6.0*(AJ/AM)**3
      GOTO 8
 7    B = 2.0*(1.0-AJ/AM)**3
 8    A = (SIN(DE*AJ)/AJ)*COS((AL-AQ)*AJ)*B
      AS(L) = AS(L)+A
      AJ = -AJ
C      IF(AJ)8,9,9
      IF(AJ .LT. 0) GOTO 8
```

```
      IF(AJ .GE. 0) GOTO 9
 9    CONTINUE
      AS(L) = AS(L)/(2.0*P)
 10   CONTINUE
      DO 11 J = 1, NL1
      WS(J) = 0.0
      AJ = J-1
      DO 111 L = 1, 12
      AL = L-1
      AL = AL*P/11.0
      W = AS(L)*COS(AJ*AL)
      WS(J) = WS(J)+W
 111  CONTINUE
 11   CONTINUE
      WS(NL1) = WS(NL1)/2.0
      SU = 0.0
      DO 12 J = 1, NL1
      SU = SU+WS(J)
 12   CONTINUE
      ST = 2.0*SU-WS(1)
      DO 13 J = 1, NL1
      WS(J) = WS(J)/ST
 13   CONTINUE
 14   WRITE (6,15)
 15   FORMAT(' WEIGHTS:')
      WRITE (6,30)(WS(J),J = 1,NL1)
 30   FORMAT(/12F6.2)
      DO 16 L = 1, 7
      AL = FLOAT(L-1)*P/6.0
      AS(L) = WS(1)
      DO 161 J = 2, NL1
      AJ = J-1
      AS(L) = AS(L)+WS(J)*COS(AJ*AL)*2.0
 161  CONTINUE
 16   CONTINUE
      DO 17 L = 1, 7
      AS(L) = 1.0-AS(L)
 17   CONTINUE
      AJS = -1.0
      DO 19 J = 1, 12
      V(J) = 0.0
      AJS = -AJS
      AJ = FLOAT(J-1)*P/6.0
      DO 18 I = 1,5
      V(J) = V(J)+COS(FLOAT(I)*AJ)/AS(I+1)
 18   CONTINUE
      V(J) = V(J)/6.0+AJS/(AS(7)*12.0)
      V(J+12) = V(J)
 19   CONTINUE
      WRITE (6,20)
 20   FORMAT(' GAIN:')
      WRITE (6,30)(AS(J),J = 1,7)
      WRITE (6,21)
 21   FORMAT(' CORRECTION FACTORS:')
      WRITE (6,30)(V(J),J = 1,12)
C     COMPUTE FILTERED SERIES FD FROM DA
      NTA = NT-NL
      DO 22 J = NL1, NTA
      FD(J) = WS(1)*DA(J)
      DO 221 L = 2,NL1
      K1 = J+L-1
      K2 = J-L+1
      FD(J) = FD(J)+WS(L)*(DA(K1)+DA(K2))
 221  CONTINUE
 22   CONTINUE
      DO 24 K = 1, 2
      DO 241 L = 3, 5
      AT(L,K) = 0.0
      NT1 = NT-12*(K-1)
      M = NT1-2*L
      DO 23 J = M, NT1
```

```
      AT(L,K) = AT(L,K)+DA(J)
23    CONTINUE
      AT(L,K) = AT(L,K)/FLOAT(2*L+1)
241   CONTINUE
24    CONTINUE
      K = NT-6
      M = NT-11
      S1 = 0.0
      S2 = 0.0
      DO 25 J = M, NT
      S1 = S1+ABS(DA(J)-FD(K))
      S2 = S2+ABS(DA(J-12)-FD(K-12))
25    CONTINUE
      S1 = S1/S2
      DO 26 L = 3, 5
      J = NT-L
      K = J-12
      FD(J) = AT(L,1)-S1*(AT(L,2)-FD(K))
26    CONTINUE
      J = NT-3
      D1 = 2.0*(FD(J)-FD(J-4))+(FD(J-1)-FD(J-3))
      D1 = D1/10.0
      M1 = NT-7
      M2 = NT-3
      D2 = 0.0
      DO 27 J = M1,M2
      D2 = D2+FD(J)
27    CONTINUE
      D2 = D2/5.0
      M2 = NT-2
      DO 28 J = M2, NT
      FD(J) = D2+FLOAT(J-M2+3)*D1
28    CONTINUE
      DO 29 J = NL1,NT
      FD(J) = DA(J)-FD(J)
29    CONTINUE
      RETURN
      END
```

Convert Octave arrays to season format: datamtoseason.m

```
% script to convert Octave data to input for
% season, i.e. a
% line giving: data description (for plot title)
% line giving:(format)   N(obs)   T(season fundamental)
% data description in: plot_title
% data is in         : odat
%

T = 12;
N = max(size(odat));

ffile= fopen('seas.dat','w');
fprintf(ffile, '%s\n', plot_title);
fprintf(ffile, '%s     %i    %i.0\n', '(12F8.0)', N, T);
nr = N/12;
nl = N- 12*nr;

for k = 1:nr;
  for j = 1:T;
    fprintf(ffile, '%8.2f', odat(12*(k-1)+j));
  endfor
  fprintf(ffile, '\n');
endfor
if(nl> 0)
  for j = 1:nl;
    fprintf(ffile, '%8.2f ', odat(12*(k-1)+j));
  endfor
  fprintf(ffile, '\n');
endif
fclose (ffile);
```

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